NOTES ON QUASI-FREE ALGEBRAS

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Abstract. These are the lecture notes from the course MATH 7350: Algebraic Differential Algebra given at Cornell in August-December 2009. The notes cover most of the content of the famous paper [CQ95a] of Cuntz and Quillen, which introduced the notion of a quasi-free algebra. The notes also include an introductory section on Morita theory and Hochschild (co)homology. Some exercises and examples are also included.

1. Introduction

Our basic objects of study are algebras, which is to say, rings which are also vector spaces. We will work over a field k (usually $\mathbb{C}$). Our algebras are not assumed to be finite-dimensional. It is difficult to say very much about algebras at this level of generality, so people usually take one of two approaches: either putting a topology on the algebra, or assuming some sort of finiteness condition.

1.1. **Topological approach.** Introduce a topology (for example, via a norm). This leads to notions like C^* algebras or Banach algebras. An example is $C^\infty(M)$ where M is a manifold. In this setting, Connes developed a lot of constructions of differential geometry for arbitrary C^* algebras by generalising the corresponding notions for $C^\infty(M)$. The kind of things to generalise are topological or differential invariants of M, for example de Rham cohomology or differential forms on M. This field is sometimes referred to as the study of noncommutative manifolds. For more information, see [Con94].

1.2. **Algebraic approach.** Parallel to the topological approach, instead impose some kind of finiteness condition, for example Noetherianness. For commutative rings, the study of finitely-generated commutative k–algebras is algebraic geometry. For general rings, this subject is usually known as "ring theory". A good reference is [MR01].

Methods introduced by Connes in the toplogical setting were developed in the algebraic setting by Cuntz and Quillen in their paper [CQ95a], and elsewhere. We intend to study this paper.

The "differential algebra" in the title of the course refers to the fact that we are studying things like differential forms, and the "algebraic" refers to the fact that we are in the algebraic setting, rather than the topological setting. This explains the title.

1.3. **Topics to be studied.** We intend to cover the following topics.

(1) Morita theory.
(2) Hochschild (co)homology and deformations.

(3) Working through the paper [CQ95a], with examples.

1.4. **Acknowledgements.** The idea of generalized representations (Sections 5.1 and 5.2) is due to Yuri Berest. The results of Sections 5.2 and Proposition 7.9 are original, as far as we are aware. Sources for the other results are given in the text. Thanks to Youssef El Fassy Fihry, George Khachatryan and Tomoo Matsumura for attending the course and making many helpful comments. Any errors in the text are the responsibility of the author.

2. Morita theory

We work over a field k, usually $k = \mathbb{C}$. Unadorned tensor products are usually over k.

2.1. **Basic definitions.** A k–**algebra** is a k–vector space A together with a bilinear multiplication $A \otimes_k A \to A$ which is associative and which has a unit element 1_A. Another way of saying this is that an algebra is a ring which is also a vector space, such that the multiplication is bilinear.

Examples 2.1. *Some algebras:*

- k.
- *The free algebra* $k\langle x_1, x_2, \ldots, x_n\rangle$.
- *Path algebra of a quiver,* kQ.
- $M_n(A)$*, where* A *is an algebra.*

If A is an algebra, a **left** A–**module** M is a vector space M equipped with a bilinear map $A \otimes M \to M$, written $a \otimes m \mapsto am$, and satisfying the axioms $1_A m = m$ for all $m \in M$, and $a(bm) = (ab)m$ for all $a, b \in A$ and all $m \in M$.

A **map** of modules (also called an A–**map** or **homomorphism**) is a linear map $f : M \to N$ which satisfies $f(am) = af(m)$ for all $a \in A$ and all $m \in M$.

The category of all left A–modules and module maps will be denoted $A - \mathrm{Mod}$. It is always abelian.

We have the analogous notion of a **right** A–**module** and the category $\mathrm{Mod} - A$. (A right A–module M has a map $M \otimes A \to M$ and satisfies the axiom $(ma)b = m(ab)$ for all $m \in M$ and all $a, b \in A$)

A module is also known as a **representation** of A.

Given two algebras A and B, an $A - B$–**bimodule** is a vector space M which is a left A–module and a right B–module, and which satisfies the additional axiom that $(am)b = a(mb)$ for all $a, b \in A$ and all $m \in M$.

An $A - A$–bimodule is often called an A–bimodule. The category of A–bimdoules will be denoted $A - \mathrm{Bimod}$.

Definition 2.2. *If* A *is an algebra with multiplication* $\cdot$*, the* opposite algebra A^{op} *is defined as the vector space* A *together with the multiplication* $\circ$ *given by* $a \circ b := b \cdot a$.

Exercise 2.3. *The following exercise shows that* $\mathrm{Mod} - A$ *and* $A - \mathrm{Bimod}$ *are special cases of* $A - \mathrm{Mod}$.

(1) *Show that a right A–module is the same thing as a left A^{op}–module.*

(2) *Show that an A–bimodule is the same thing as a left $A \otimes_k A^{op}$–module.*

Note that in the above exercise, we used the fact that if A and B are algebras then there is a natural algebra structure on the tensor product $A \otimes B$. This can easily be checked.

A left A–module M is **finitely-generated**, or **f.g.** for short, if there exist $m_1, \ldots, m_n \in M$ such that $M = \sum_{i=1}^{n} Am_i$. We will write $A - \mathrm{mod}$ and $\mathrm{mod} - A$ for the full subcategories of $A - \mathrm{Mod}$ and $\mathrm{Mod} - A$ consisting of the f.g. modules. We usually study only f.g. modules when we insist that our algebras satisfy conditions like Noetherianness (see [MR01, Chapter 0]). But caution! In general, $A - \mathrm{mod}$ need not be an abelian category.

2.2. **Morita Theory.** Morita theory addresses the question of when two algebras A and B have equivalent categories of modules.

Definition 2.4. *Given algebras A and B, we say that A and B are* Morita equivalent *if $A - \mathrm{Mod}$ is equivalent to $B - \mathrm{Mod}$.*

Before studying Morita equivalence, we first give some examples.

Definition 2.5. *Let $\mathcal{C}$ be a category. The* centre *of $\mathcal{C}$ is the set $Z(\mathcal{C}) = \mathrm{Nat}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$ of natural transformations from the identity functor on $\mathcal{C}$ to itself.*

It is easy to see that the centre of an additive category is a ring. (Exercise).

Proposition 2.6. *If A is a ring then $Z(A - \mathrm{Mod})$ is isomorphic to $Z(A)$.*

Proof. To give a natural transformation $\phi : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is the same as giving a map $\phi_M : M \to M$ for all A–modules M, such that if $f : M \to N$ is any map then the following square commutes.

$$\begin{tikzcd} M \arrow[r, "\phi_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "\phi_N"] & N \end{tikzcd}$$

Given such a ϕ, we have in particular the A–module map $\phi_A : A \to A$. This satisfies $\phi_A(a) = a\phi_A(1)$, so it is determined by $\phi_A(1) \in A$. If M is an arbitrary A–module and $m \in M$, then there is a map $A \to M$ g iven by $a \mapsto am$. So $\phi_M(m) = \phi_A(1)m$ and hence ϕ is fully determined by $\phi_A(1)$. Thus, the map $Z(A - \mathrm{mod}) \to A$ given by $\phi \mapsto \phi_A(1)$ is injective. Also, if $b \in A$ then $\cdot b : A \to A$ is an A–module map, where $\cdot b$ denotes right multiplication by b. This implies that $\phi_A(1)b = b\phi_A(1)$ for all $b \in A$, and therefore $\phi_A(1) \in Z(A)$. It is now easy to finish the proof by checking that $Z(A - \mathrm{mod}) \to Z(A)$ is surjective, and that it respects the ring structure given by the above exercise. $\square$

Corollary 2.7. *If A and B are commutative and Morita equivalent, then A and B are isomorphic.*

In geometric language, the corollary says (in particular) that an affine variety is determined by the category of quasicoherent sheaves on it.

It is now natural to ask whether it is possible for two nonisomorphic rings to be Morita equivalent. The answer is yes. For example, take $A = k$, $B = M_2(k)$. Then A and B are not isomorphic because A is commutative and B is not. But $A - \mathrm{Mod}$ is equivalent to $B - \mathrm{Mod}$. To see this, we need to use the fact that every B–module is a direct sum of copies of the module k^2 of column vectors. This follows from some elementary algebra, although proving it from scratch might not be straightforward. Given this fact, we may define a functor $A - \mathrm{mod} \to B - \mathrm{mod}$ sending $k^{\oplus n}$ to $(k^2)^{\oplus n}$. Because both $A - \mathrm{mod}$ and $B - \mathrm{mod}$ are semisimple categories with a unique simple object whose endomorphism ring is k, they are equivalent. Note that this doesn't actually show that $A - \mathrm{Mod}$ is equivalent to $B - \mathrm{Mod}$, but we will show later that this is the case.

2.3. **Equivalences of module categories.** We prove some lemmas about equivalences.

Lemma 2.8. *Suppose $F : A - \mathrm{Mod} \to B - \mathrm{Mod}$ is an equivalence. Suppose M is a f.g. A–module. Then $F(M)$ is f.g.*

Proof. We just need to show that being finitely-generated is a categorical property. This is true because we can express finite generation as follows:

A module M is finitely-generated if and only if for every surjection $\pi : \bigoplus_{i\in I} N_i \twoheadrightarrow M$ there exists a finite subset $J \subset I$ such that the restriction of π to $\bigoplus_{j\in J} N_j$ is surjective.

It is an exercise to show that the above property is equivalent to M being f.g, which finishes the proof. $\square$

The next lemma describes an arbitrary equivalence.

Lemma 2.9. *Suppose $F : A - \mathrm{Mod} \to B - \mathrm{Mod}$ is an equivalence. Then there exists a unique bimodule ${}_BQ_A$ such that there is an isomorphism of functors $F \cong H$, where $H(M) = {}_BQ_A \otimes_A M$ for all A–modules M.*

Proof. The following proof by Ginzburg and Boyarchenko is taken from the notes [Gin05].

We will take $Q = F(A)$. We need to show that this is a $B - A$–bimodule. By definition, $F(A)$ is a left B–module. To show that it is a right A–module, for $a \in A$ we consider the map $\cdot a : A \to A$ given by right multiplication by a. Then $F(\cdot a)$ is a B–map. This makes $F(A)$ into a right A–module, and it is straightforward to check that the actions of A and B commute, so that $F(A)$ is a bimodule.

To define a natural transformation from H to F, we need to define a $\phi_M : H(M) \to F(M)$ for every $M \in A - \mathrm{Mod}$. Given $x \otimes m \in H(M)$, with $x \in F(A)$ and $m \in M$, we define $\phi_M(x \otimes m) = F(\rho_m)(x)$ where $\rho_m : A \to M$ is defined by $\rho_m(a) = am$.

It is necessary to check that this is well-defined, which amounts to checking that $xa \otimes m$ and $x \otimes am$ have the same image for $a \in A$. Also, if $b \in B$ then $\phi_M(bx \otimes m) = F(\rho_m)(bx) = bF(\rho_m)(x)$ since $F(\rho_m)$ is a B–map. This shows that ϕ_M is a B–map as well.

Now we need to show that the maps ϕ_M are natural. That is, if $\lambda : M \to N$ is an A–map, we need to show that the following square commutes.

$$\begin{tikzcd} H(M) \arrow[r, "H(\lambda)"] \arrow[d, "\phi_M"'] & H(N) \arrow[d, "\phi_N"] \\ F(M) \arrow[r, "F(\lambda)"] & F(N) \end{tikzcd}$$

For $x \otimes m \in H(M)$, we have $F(\lambda)\phi_M(x \otimes m) = F(\lambda)F(\rho_m)(x \otimes m) = F(\lambda\rho_m)(x) = F(\rho_{\lambda(m)})(x) = \phi_N H(\lambda)(x \otimes m)$. This proves the naturality.

Thus, $\phi = (\phi_M)$ is a natural transformation $H \to F$ of functors $A - \text{Mod} \to B - \text{Mod}$. We need to show that it is an isomorphism. Let $M \in A - \text{Mod}$. Then there is an exact sequence

$$A^{\oplus J} \to A^{\oplus I} \to M \to 0$$

for some (possibly infinite) sets I and J. We form this by having some free module surject onto M, and then having another free module surject onto the kernel of $(A^{\oplus I} \to M)$.

Naturality, together with the fact that our functors are equivalences, yields a diagram:

$$\begin{tikzcd} H(A^{\oplus J}) \arrow[r] \arrow[d] & H(A^{\oplus I}) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ F(A^{\oplus J}) \arrow[r] & F(A^{\oplus I}) \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

It is easy to check that for any indexing set I, the map $\phi_{A^{\oplus I}}$ is an isomorphism, since ϕ_A is. Therefore, the first two vertical maps in the above diagram are isomorphisms. The Five Lemma then implies that the map $\phi_M : M \to M$ is also an isomorphism.

Finally, for the uniqueness of Q, observe that if R is an $B - A$–bimodule such that $F(M) = R \otimes_A M$ for all M, then $R \cong F(A)$. There is a little more work to be done to check that this is really an isomorphism of bimodules. $\square$

Corollary 2.10. *Two rings A and B are Morita equivalent if and only if there exist bimodules ${}_BQ_A$ and ${}_AR_B$ such that $Q \otimes_A R \cong B$ as B–bimodules and $R \otimes_B Q \cong A$ as A–bimodules.*

Proof. If such Q, R exist, define $F(M) = Q \otimes_A M$ and $G(N) = R \otimes_B N$. Then F and G are a pair of inverse equivalences between $A - \text{Mod}$ and $B - \text{Mod}$. Conversely if $F : A - \text{Mod} \to B - \text{Mod}$ and $G : B - \text{Mod} \to A - \text{Mod}$ are inverse equivalences, then by the above lemma, we have

$$A \cong GF(A) \cong G(B) \otimes_B F(A)$$

and

$$B \cong FG(B) \cong F(A) \otimes_A G(B)$$

as required. (That these are really bimodule isomorphisms follows from the fact that $\mathrm{id} \cong GF$ and $\mathrm{id} \cong FG$ are isomorphisms of functors.) $\square$

We get the following not-obvious-from-the-definition corollary.

Corollary 2.11. *If A and B are rings then $A - \mathrm{Mod}$ is equivalent to $B - \mathrm{Mod}$ if and only if $\mathrm{Mod} - A$ is equivalent to $\mathrm{Mod} - B$.*

Proof. The condition in Corollary 2.10 is symmetric in A and B. $\square$

2.4. **Progenerators.** Consider an equivalence $F : A - \mathrm{Mod} \to B - \mathrm{Mod}$. The B–module ${}_BF(A)$ is:

- projective.
- finitely-generated (by Lemma 2.8).
- a *generator*.

Definition 2.12. *A module ${}_AM$ is called a* generator *if for all nonzero $f : X \to Y$ in $A - \mathrm{Mod}$, there exists some $h : M \to X$ with $fh \neq 0$.*

The above definition makes sense in any additive category. It is easy to see that ${}_AA$ is a generator, therefore so is ${}_BF(A)$. Furthermore, ${}_BF(A)$ is f.g. because A is, and we have shown that finite-generation is preserved by equivalences. Thus, ${}_BF(A)$ is a *progenerator*.

Definition 2.13. *A* progenerator *in a module category is an object which is a finitely-generated projective generator.*

We need other ways to characterise being projective and being a generator.

Proposition 2.14. *A module ${}_AM$ is a generator if and only if the evaluation map*

$$M \times \mathrm{Hom}_A(M, A) \to A$$

is surjective.

Proof. This proof is taken from [Row08, section 25A].

If ${}_AM$ is a generator, set I to be the image of the evaluation map $M \times \mathrm{Hom}_A(M, A) \to A$. Then I is a two-sided ideal of A. If $I \neq A$, the consider $\pi : A \to A/I$. There is no $h : M \to A$ with $h(M) \neq 0$, a contradiction. Therefore, $I = A$.

Conversely, if the evaluation map is surjective then there exist $g_i : M \to A$ and $m_i \in M$ with $\sum g_i(m_i) = 1_A$. Now let $f : X \to Y$ be any nonzero map of left A–modules. Suppose $f(x) \neq 0$. Define $\lambda_i : M \to X$ by $\lambda_i(m) := g_i(m)x$. Then

$$f \sum \lambda_i(m_i) = f \sum g_i(m_i)x = f(x) \neq 0$$

and hence $f\lambda_i(m_i) \neq 0$ for some i, so $f\lambda_i \neq 0$. $\square$

Exercise 2.15. *Now show that M is a generator if and only if there exists $n \geq 1$ such that there is a surjection $M^{\oplus n} \twoheadrightarrow A$.*

Lemma 2.16 (Dual basis lemma)**.** *A module ${}_AM$ is f.g. projective if and only if there exist $x_1, \dots x_n \in M$ and $\varphi_1, \dots, \varphi_n : M \to A$ such that*

$$m = \sum_{i=1}^{n} \varphi_i(m) x_i$$

for all $m \in M$.

Proof. If M is a f.g. projective module, let $i : M \to A^n$ and $\pi : A^n \to M$ be a splitting, that is, $\pi i = \mathrm{id}_M$. Define φ_i to be the composition of $M \to A^n$ with the i^{th} projection. Then $i : m \mapsto (\varphi_1(m), \dots, \varphi_n(m))$. Let $x_i = \pi(0, \dots, 1, 0, \dots, 0)$, the image under π of a vector with 1 in the i^{th} place and zeroes elsewhere. Then $m = \sum \varphi_i(m) x_i$ for all $m \in M$.

Conversely, given a dual basis $\{\varphi_i\}, \{x_i\}$, define $M \to A^n$ via $m \mapsto (\varphi_1(m), \dots, \varphi_n(m))$ and define $A^n \to M$ via $(0, \dots, 1, 0, \dots, 0) \mapsto x_i$. This gives a splitting, so M is projective. $\square$

Corollary 2.17. *${}_AM$ is f.g. projective if and only if the map*

$$\mathrm{Hom}_A(M, A) \times A \to \mathrm{End}_A(M)$$

defined by $\psi \otimes m \mapsto (n \mapsto \psi(n)m)$ is surjective.

Proof. This is just a restatement of the existence of a dual basis. $\square$

We have seen that an equivalence of module categories gives rise to a progenerator. Now we can show the opposite.

Theorem 2.18. *Let ${}_AQ$ be a progenerator for $A-\mathrm{Mod}$. Let $B = \mathrm{End}_A({}_AQ)^{op}$. Then A is Morita equivalent to B.*

Proof. Let $Q^* = \mathrm{Hom}_A({}_AQ, A)$. Then Q^* is a left B–module if we define, for $\psi \in Q^*$, $a \in Q$ and $b \in B$, $b \cdot \psi(q) = \psi(qb)$. We can also make Q^* into a right A–module by defining $(\psi \cdot a)(q) = \psi(q)a$ for $a \in A$. It is an exercise to check that this is a well-defined bimodule structure.

We wish to show that the natural maps

$$Q \otimes_B Q^* \to A \qquad q \otimes \psi \mapsto \psi(q)$$

$$Q^* \otimes_A Q \to B \qquad \psi \otimes q \mapsto (q' \mapsto \psi(q')q)$$

are isomorphisms of bimodules. It is again an exercise to check that these are well-defined bimodule maps. If we can show that they are isomorphisms, it will follow that the functors $Q \otimes_A (-)$ and $Q^* \otimes_B (-)$ give an equivalence between $A - \mathrm{Mod}$ and $B - \mathrm{Mod}$.

To show that the map $Q \otimes_B Q^* \to A$ is an isomorphism, we first note that the map is surjective because Q is a generator. Now suppose $z_i \in Q$ and $\zeta_i \in Q^*$ and $\sum_{i=1}^n z_i \otimes \zeta_i \mapsto 0$. Since Q is a generator, there exist $\varphi_i : Q \to A$, $1 \leq i \leq N$ and $q_i \in Q$, such that $\sum_{i=1}^N \varphi_i(q_i) = 1_A$. Then

$$\sum_{i=1}^n z_i \otimes \zeta_i = \sum_{i=1}^n \sum_{j=1}^N \varphi_j(q_j) z_i \otimes \zeta_i = \sum_{i,j} b_{ij}(z_i) \otimes \zeta_i$$

where we define $b_{ij}(x) = \varphi_j(x) z_i$ for $x \in Q$. Regarding b_{ij} as an element of B acting on Q from the right, we have

$$\sum_{i,j} b_{ij}(q_j) \otimes \zeta_i = \sum_{i,j} q_j \cdot b_{ij} \otimes \zeta_i = \sum_j q_j \otimes \sum_i b_{ij} \cdot \zeta_i$$

Now, for $q \in Q$, $(\sum_i b_{ij} \cdot \zeta_i)(q) = \sum_i \zeta_i(q b_{ij}) = \sum_j \zeta_i(b_{ij}(q)) = \sum_i \zeta_i(\varphi_j(q) z_i) = \varphi_j(q) \sum_i \zeta_i(z_i) = 0$. Therefore, $\sum_{i=1}^n z_i \otimes \zeta_i = 0$ as required.

The second map is surjective because Q is a f.g. projective module. We need to show that it is injective. Suppose $\sum_{i=1}^n \lambda_i \otimes q_i \mapsto 0$ for some $\lambda_i \in Q^*$ and $q_i \in Q$. Then $\sum_i \lambda_i(q) q_i = 0$ for all $q \in Q$. Now, since Q is projective, by the Dual Basis Lemma there exist $\varphi_i : Q \to A$, $1 \leq i \leq M$, and $x_i \in Q$ with $\sum_{j=1}^M \varphi_j(q) x_j = q$ for all $q \in Q$. We have

$$\sum_i \lambda_i \otimes q_i = \sum_i \lambda_i \sum_{j=1}^M \varphi_j(q_i) x_j = \sum_j (\sum_i \lambda_i \cdot \varphi_j(q_i)) \otimes x_j.$$

We show that $\sum_i \lambda_i \cdot \varphi_j(q_i) = 0$ for all j. Indeed, if $z \in Q$ then $\sum_i \lambda_i \cdot \varphi_j(q_i) : z \mapsto \sum_i \lambda_i(z) \varphi_j(q_i) = \varphi_j(\sum_i \lambda_i(z) q_i) = 0$ as required. $\square$

Corollary 2.19 (Morita)**.** *Two rings A and B are Morita equivalent if and only if there is a progenerator ${}_AQ$ for $A-\mathrm{Mod}$ such that $B \cong \mathrm{End}_A({}_AQ)^{op}$.*

Proof. We have shown one direction. Conversely, if A is Morita equivalent to B, let $G : B-\mathrm{Mod} \to A-\mathrm{Mod}$ be an equivalence. Then $B^{op} \cong \mathrm{End}_B({}_BB) \cong \mathrm{End}_A({}_AG(B))$ and we have already explained why ${}_AG(B)$ is a progenerator. $\square$

Corollary 2.20. *Two rings A and B are Morita equivalent if and only if there exists $n \geq 1$ and an idempotent $e \in M_n(A)$ with $M_n(A) e M_n(A) = M_n(A)$ such that $B \cong e M_n(A) e$.*

Proof. Let ${}_AQ$ be a progenerator for $A-\mathrm{Mod}$ such that $B \cong \mathrm{End}_A({}_AQ)^{op}$. Let $\pi_Q : A^n \to Q$ and $i_Q : Q \to A^n$ be such that $\pi_Q i_Q = \mathrm{id}_Q$. Define $e \in \mathrm{End}_A(A^n)$ by $e = i_Q \pi_Q$. Then Q is isomorphic to the image $e(A^n)$ of e.

We have $B^{op} = \mathrm{End}_A({}_AQ) \cong \mathrm{End}_A(e(A^n)) \cong e\mathrm{End}_A(A^n)e$ (it is straightforward to check the last equality). But $\mathrm{End}_A(A^n)$ can be identified with $M_n(A)^{op}$ via the action of $M_n(A)$ on A^n by multiplication on the right. So $B^{op} \cong e M_n(A)^{op} e \cong (e M_n(A) e)^{op}$ whence $B \cong e M_n(A) e$.

Now we need to show that $M_n(A) e M_n(A) = M_n(A)$, ie. $\mathrm{End}_A(A^n) e \mathrm{End}_A(A^n) = \mathrm{End}_A(A^n)$.

Let $\lambda_k : A \to A^n$ and $\pi_k : A^n \to A$ be the k^{th} canonical insertion and projection, respectively. Let $\ell_i : Q \to Q^N$ and $p_i : Q^N \to Q$ likewise denote the i^{th} canonical insertion and projection. Since Q is a generator, there exist $\psi_1, \dots, \psi_N : Q \to A$ and $q_1, \dots, q_N \in Q$ with $\sum_{i=1}^N \psi_i(q_i) = 1_A$. Define $\gamma : A \to Q^N$ by $\gamma(a) = (aq_1, \dots, aq_n)^T$ and define $\rho : Q^N \to A$ by $\rho(r_1, \dots, r_N) = \sum \psi_i(r_i)$. Then $\rho\gamma = \mathrm{id}_A$.

We then have:

$$\begin{aligned}
\mathrm{id}_{A^n} &= \sum_{i=1}^n \pi_i \lambda_i \\
&= \sum_{i=1}^n \pi_i \rho\gamma\lambda_i \\
&= \sum_{i=1}^n \pi_i \rho (\sum_{j=1}^N \ell_j p_j) \gamma \lambda_i \\
&= \sum_{i,j} \pi_i \rho \ell_j (\pi_Q i_Q)^2 p_j \gamma \lambda_i \\
&= \sum_{i,j} (\pi_i \rho \ell_j \pi_Q) e (i_Q p_j \gamma \lambda_i)
\end{aligned}$$

Which shows that $\mathrm{End}_A(A^n) e \mathrm{End}_A(A^n) = \mathrm{End}_A(A^n)$ contains id_{A^n}. Since it is a two-sided ideal, it is therefore the whole of $\mathrm{End}_A(A^n)$ as desired.

Conversely, suppose $B \cong eM_n(A)e$. We show that B is Morita equivalent to A by proving the following two facts for any ring A.

(1) A is Morita equivalent to $M_n(A)$.
(2) If $e \in A$ is an idempotent and $AeA = A$ then A is Morita equivalent to eAe.

To prove the first statement, we have that A^n is a progenerator in $A - \mathrm{Mod}$, and $\mathrm{End}_A(A^n) \cong M_n(A)^{op}$, so A is Morita equivalent to $M_n(A)$ by Corollary 2.19.

To prove the second statement, We take $Q = Ae$. Then Q is finitely-generated since it is generated over A by e. It is also projective because $A = Ae \oplus A(1-e)$. We must show that Q is a generator. We have $\mathrm{Hom}_A(Ae, A) \cong eA$ (isomorphism of abelian groups). This isomorphism can be defined by mapping $f : Ae \to A$ to $f(e)$. So Ae is a generator if and only if the evaluation map $Ae \times eA \to A$ is surjective. But this evaluation map is just multiplication, so Ae is a generator if and only if $AeA = A$. Thus, Ae is a generator.

Now, $\mathrm{End}_A(Ae) \cong eA^{op}e \cong (eAe)^{op}$ (isomorphism of rings) via $f \mapsto f(e)$. It is an exercise to check that this really is a ring isomorphism. Thus, we obtain that A is equivalent to eAe as required. □

Corollary 2.20 is a version of Morita's Theorem which is commonly used in practice to show that some property is Morita invariant. It is quite hard to find in textbooks; one good reference is [MR01, 3.5.6].

Remarks 1. Some remarks:

(1) If $A-\mathrm{Mod}$ is equivalent to $B-\mathrm{Mod}$ then $A-\mathrm{mod}$ is equivalent to $B-\mathrm{mod}$. This is because tensoring with a progenerator preserves the property of being f.g, because progenerators are by definition f.g. But $A-\mathrm{mod}$ being equivalent to $B-\mathrm{mod}$ does not imply that $A-\mathrm{Mod}$ is equivalent to $B-\mathrm{Mod}$. This is because $R-\mathrm{mod}$ need not be an abelian category. However, if A and B are Noetherian algebras, then we do get the second implication. Usually, we deal with Noetherian algebras.
(2) The version of Morita theory presented here is the simplest kind. There are versions for other mathematical objects, for example derived categories.
(3) A property which is invariant under Morita equivalence is called a *Morita invariant*. We have seen that $Z(A)$ is a Morita invariant. Other Morita invariants include $K_0(A)$ and $HH_*(A)$ (to be defined below). In attempts to generalise algebraic geometry to noncommutative rings, it is believed that any "geometric" property should be Morita invariant (see [Gin05, Section 2.2]). Thus, Morita invariance is rather important.

Exercises 2.21. Exercises on Morita theory.

(1) Complete the proof of Lemma 2.8.
(2) Let A_i and B_i be rings. Show that if A_1 is Morita equivalent to B_1 and A_2 is Morita equivalent to B_2 then $A_1 \times A_2$ is Morita equivalent to $B_1 \times B_2$.
(3) If $R \cong S$ then R and S are Morita equivalent. Exhibit a progenerator ${}_SQ_R$ which realises an equivalence $R-\mathrm{Mod} \to S-\mathrm{Mod}$.
(4) Let S be a simple algebra (that is, S has no nontrivial two-sided ideals). Let G be a finite group acting linearly on S. Show that the ring of invariants S^G is Morita equivalent to the group ring $S*G$ of G with coefficients in S (the multiplication in this ring is defined by $s_1g_1 \cdot s_2g_2 = s_1g_1(s_2)g_1g_2$).
(Hint: $S * G$ is also a simple algebra.)
(5) Let R be an arbitrary algebra. Let $F : R-\mathrm{Mod} \to \mathrm{Vect}_k$ denote the forgetful functor. Show that $\mathrm{Nat}(F, F)$ is a ring, isomorphic to R.

3. Hochschild homology and cohomology

In this section we will study a homology and cohomology theory for algebras. The idea is to associate a sequence of abelian groups to an algebra A, and hopefully to obtain interesting invariants in this manner. This section is partly based on notes by Yu. Berest.

Definition 3.1. *Given an algebra A, the* enveloping algebra *is $A^e := A \otimes_k A^{op}$.*

Recall that $A^e-\mathrm{Mod}$ is equivalent to $A-\mathrm{Bimod}$. It is sometimes useful to think of a bimodule as a left A^e–module.

Let A be a k–algebra and M an A–bimodule.

Definition 3.2. *The n^{th}* Hochschild homology *of A with coefficients in M is the vector space*

$$H_n(A, M) = \mathrm{Tor}_n^{A-\mathrm{Bimod}}(M, A).$$

The n^{th} Hochschild cohomology *of A with coefficients in M is the vector space*

$$H^n(A, M) = \mathrm{Ext}^n_{A-\mathrm{Bimod}}(A, M).$$

Aim: to calculate these, and see if they give interesting invariants.

Note that it is not immediately clear that the Tor makes sense, since we regard the category $A-\mathrm{Bimod}$ as $A^e-\mathrm{Mod}$, and the tensor product of two left R–modules doesn't make sense for a general ring R. However, in the case $R = A^e$, we have an isomorphism $R \cong R^{op}$ given by $a \otimes b \mapsto b \otimes a$. In this way, we can regard an A–bimodule M as a right A^e–module, and so we have the tensor product $M \otimes_{A^e} A$, and the definition of $H_n(A, M)$ makes sense.

Exercise 3.3. *If M and N are A–bimodules, check that $M \otimes_{A^e} N$ and $N \otimes_{A^e} M$ are naturally isomorphic, and hence conclude for all n that $\mathrm{Tor}_n^{A-\mathrm{Bimod}}(M, N) \cong \mathrm{Tor}_n^{A-\mathrm{Bimod}}(N, M)$ naturally in M and N.*

We need a projective resolution of A in the category of A–bimodules. There are many possible choices, and in fact we will use two different ones.

For an element $a_1 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}$, we often write $(a_1, \ldots, a_n)$ instead. Another traditional way of writing this is $[a_1|\cdots|a_n]$, but we will not use this notation.

We define $C_n^{bar}(A) := A^{\otimes(n+2)}$ and we define $b_n : C_n^{bar}(A) \to C_{n-1}^{bar}(A)$ by

$$b_n(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}).$$

This is clearly a bimodule map, where we give $C_n^{bar}(A)$ the structure of an A–bimodule via left and right multiplication on the first and last factors of the tensor product respectively. We want to show that

$$C_*^{bar}(A) := \cdots \to A^{\otimes(n+2)} \to A^{\otimes(n+1)} \to \cdots \to A^{\otimes 2} \to A \to 0$$

is a free resolution of A in the category $A - \mathrm{Bimod}$.

First, it is necessary to check that $C_*^{bar}(A)$ is a complex, that is, $b_{n-1}b_n = 0$. This calculation is left as an exercise. As an example, we show what happens in the lowest degree:

$$b_1 b_2(a_0, a_1, a_2) = b_1((a_0 a_1, a_2) - (a_0, a_1 a_2)) = (a_0 a_1)a_2 - a_0(a_1 a_2) = 0$$

using associativity of A. In higher degrees, the terms cancel in pairs in a similar way.

Now we prove that $C_*^{bar}(A)$ is an exact complex in degree ≥ 0. For $n \geq 2$, define $s : A^{\otimes n} \to A^{\otimes(n+1)}$ by $s(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$. We check that this is a homotopy between the identity map $C_*^{bar}(A) \to$

$C^{bar}_*(A)$ and zero. We have

$$
\begin{aligned}
(bs+sb)(a_0,\dots,a_n) &= b(1,a_0,\dots,a_n) + s\sum_{i=0}^{n-1}(-1)^i(a_0,\dots,a_ia_{i+1},\dots a_n)\\
&= (a_0,\dots,a_n) + \sum_{i=0}^{n-1}(-1)^{i+1}(1,a_0,\dots,a_ia_{i+1},\dots a_n) + \sum_{i=0}^{n-1}(-1)^i(1,a_0,\dots,a_ia_{i+1},\dots a_n)\\
&= (a_0,\dots,a_n)\\
&= (\mathrm{id}-0)(a_0,\dots,a_n)
\end{aligned}
$$

as required.

Therefore, the complex

$$\cdots \to A^{\otimes(n+2)} \to A^{\otimes(n+1)} \to \cdots \to A^{\otimes 3} \to A^{\otimes 2},$$

with differential b, is exact.

Finally, we check that each term is a projective A^e–module. Indeed, $A^{\otimes(n+2)} = A \otimes_k A^{\otimes n} \otimes_k A \cong A^e \otimes_k A^{\otimes n}$ as left A^e–modules. This is just a direct sum of (possibly infinitely many) copies of A^e, so it is a free A^e–module.

We conclude that $C^{bar}_*(A)$ is a projective (and in fact free) resolution of A in the category $A-\mathrm{Bimod}$. Thus, Hochschild homology and cohomology can be computed using the following formulas.

$$
\begin{aligned}
H_n(A,M) &= H_n(M \otimes_{A^e} C^{bar}_*(A))\\
H^n(A,M) &= H^n(\mathrm{Hom}_{A^e}(C^{bar}_*(A),M))
\end{aligned}
$$

Example 3.4. Take $A = k$. Then for all n, $A^{\otimes(n+2)}$ is isomorphic to A via $(a_0,\dots,a_{n+1}) \mapsto a_0a_1\cdots a_{n+1}$. The differential b becomes the identity map $A^{\otimes(n+2)} \to A^{\otimes(n+1)}$ if n is even, and the zero map if n is odd. Therefore, the bar resolution becomes:

$$\cdots \longrightarrow k \xrightarrow{id} k \xrightarrow{0} k \xrightarrow{id} k \xrightarrow{0} k \longrightarrow 0$$

Tensoring with M gives

$$\cdots \longrightarrow M \xrightarrow{id} M \xrightarrow{0} M \xrightarrow{id} M \xrightarrow{0} M \longrightarrow 0$$

From this we conclude that $H_0(A,M) = M$ and $H_i(A,M) = 0$ if $i > 0$. Note that we can check that this is correct because $k-\mathrm{Bimod}$ is the same as $k-\mathrm{Mod}$ as everything is k–linear, and so all higher Tors vanish.

Similarly, we calculate $H^0(A,M) = M$ and $H^i(A,M) = 0$ if $i > 0$.

3.1. **Hochschild cochain complex.** Hochschild cohomology of a bimodule M is calculated from the complex $\mathrm{Hom}_{A^e}(C^{bar}_*(A),M)$. A better way of writing the space $\mathrm{Hom}_{A^e}(C^{bar}_n(A),M)$ is $\mathrm{Hom}_{A^e}(A^e \otimes_k A^{\otimes n},M) \cong \mathrm{Hom}_k(A^{\otimes n},M)$.

Definition 3.5. $C^n(A,M) = \mathrm{Hom}_k(A^{\otimes n}, M)$ *is called the space of* Hochschild n–cochains with values in M.

The differential $d : C^n(A,M) \to C^{n+1}(A,M)$ induced from b may be computed. It is left as an exercise to verify that for $f \in C^n(A,M)$,

$$df(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_n) + \sum_{i=1}^{n}(-1)^i f(a_1,\dots,a_i a_{i+1},\dots,a_{n+1}) + (-1)^{n+1} f(a_1,\dots,a_n)a_{n+1}.$$

The complex $(C^\bullet(A,M), d)$ computes Hochschild cohomology of A with coefficients in M, and is referred to as the *Hochschild cochain complex.*

3.2. **Reduced bar complex.** Before computing some more examples, we mention another projective resolution of A in $A-\mathrm{Bimod}$. It is a quotient of the bar complex. We define it here because it is used in the paper [CQ95a]. It also illustrates that there are many different ways of calculating Hochschild (co)homology.

Definition 3.6. *Define*

$$\overline{C}_n^{bar}(A) = A \otimes_k \overline{A}^{\otimes n} \otimes_k A \cong A^e \otimes_k \overline{A}^{\otimes n}$$

where $\overline{A}$ is the vector space $A/k.1_A$.

Checking the following facts is left as an exercise: b descends to a well-defined bimodule map $\overline{C}_n^{bar}(A) \to \overline{C}_{n-1}^{bar}(A)$. Furthermore, the complex $(\overline{C}_*^{bar}(A), b)$ is contractible, using the same s as before. Therefore, $\overline{C}_*^{bar}(A)$ is a projective resolution of A in the category $A-\mathrm{Bimod}$.

Definition 3.7. $\overline{C}_*^{bar}(A)$ *is called the* reduced bar complex *of A.*

Similarly, we have the reduced Hochschild cochain complex of a module M, whose n^{th} term is $\mathrm{Hom}_k(\overline{A}^{\otimes n}, M)$. Later, we will interpret $A \otimes_k \overline{A}^{\otimes n}$ as the space of noncommutative n–forms on A, so the reduced bar complex will give a relationship between noncommutative forms and Hochschild theory.

Example 3.8. For $A = k$, we have $\overline{A} = 0$. So the reduced bar complex of A has just one term, in degree 0, and we therefore obtain $H_i(A,M) = M$ if $i = 0$ and $H_i(A,M) = 0$ if $i > 0$, as before.

3.3. **Morita invariance.** Now we prove Morita invariance, which is very useful for calculating Hochschild homology and cohomology.

Theorem 3.9. *If A and B are algebras and M is an A–bimodule, and ${}_BP_A$ and ${}_AQ_B$ are bimodules such that $P \otimes_A Q \cong B$ as B–bimodules, and $Q \otimes_B P \cong A$ as A–bimodules, then*

$$H^n(A,M) \cong H^n(B, P \otimes M \otimes Q)$$

$$H_n(A,M) \cong H_n(B, P \otimes M \otimes Q)$$

for all n.

Proof. We define a functor $A - \text{Bimod} \to B - \text{Bimod}$ by $M \mapsto P \otimes M \otimes Q$. This is an equivalence because we can define an inverse by $N \mapsto Q \otimes N \otimes P$. Since it is an equivalence of module categories, it preserves Ext and Tor. Furthermore, we have $P \otimes A \otimes Q \cong A$ and $Q \otimes B \otimes P \cong B$ by the hypothesis. ☐

Corollary 3.10. *If A is Morita equivalent to B then for all n,*

$$H^n(A, A) \cong H^n(B, B)$$

$$H_n(A, A) \cong H_n(B, B)$$

Example 3.11. Regarding k^n as a space of row vectors, we have ${}_k k^n_{M_n(k)}$. Clearly, k^n is a progenerator in $k - \text{Mod}$, so this k^n realises a Morita equivalence. The inverse equivalence is realised by tensoring with the module $\text{Hom}_k(k^n, k)$ of column vectors. The equivalence of bimodules $k - \text{Bimod} \to M_n(k) - \text{Bimod}$ which we obtain from Theorem 3.9 sends k^p to $M_n(k)^p$. Thus, every f.g. $M_n(k)$–bimodule is isomorphic to $M_n(k)^p$ for some p, and from Theorem 3.9 we obtain

$$H_i(M_n(k), M_n(k)^p) \cong \begin{cases} k^p & i = 0 \\ 0 & i > 0. \end{cases}$$

3.4. **Low-dimensional calculations.** In this section, we calculate $H_i(A, M)$ and $H^i(A, M)$ for $i = 0, 1, 2$ and certain choices of A and M, in order to get a flavour of what Hochschild homology and cohomology are like.

First, we look at $H_0(A, M)$. By definition, this is $M \otimes_{A^e} A$. It is not immediately clear what this is. It can be computed using the bar resolution. The relevant part of the bar resolution is

$$A^{\otimes 3} \to A^{\otimes 2} \to 0$$

which we regard as

$$A^e \otimes_k A \to A^e \to 0.$$

Then $M \otimes_{A^e} A$ is $M/\text{im}b$ where $b : M \otimes_k A \to M$ is the map induced from the Hochschild differential. We can calculate explicitly what this is. If $m \otimes a \in M \otimes_k A$, then $m \otimes a$ corresponds to $m \otimes (1, a, 1) \in M \otimes_k A^{\otimes 3}$. This is taken by b to $m \otimes ((a, 1) - (1, a)) \in M \otimes_{A^e} A^e$. According to the definition of the right A^e–action on M, this corresponds to $ma - am \in M$. Thus, $M \otimes_{A^e} A$ is $M/[A, M]$ where $[A, M]$ is the sub*space* of M consisting of $\{ma - am : m \in M, a \in A\}$.

Now we calculate $H^0(A, M)$. This is the kernel of $d : C^0(A, M) \to C^1(A, M)$ given by $d(m) = am - ma$. Therefore

$$H^0(A, M) = \{m \in M : am = ma \quad \forall a \in A\}.$$

In particular, $H^0(A, A) = Z(A)$.

Next, the map $d : C^1(A, M) \to C^2(A, M)$ is given by $df(a, b) = af(b) - f(ab) - f(a)b$. The kernel of this map is

$$\{f : A \to M : f(ab) = af(b) + f(a)b \quad \forall a, b \in A\}$$

which is also known as $\mathrm{Der}_A(A, M)$. As computed above, the image of $d : C^0(A, M) \to C^1(A, M)$ consists of those $f : A \to M$ of the form $f(m) = am - ma$ for some fixed $m \in M$. This is the space of inner derivations of M, denoted $\mathrm{Inn}_A(M)$. Thus,

$$H^1(A, M) = \mathrm{Der}_A(A, M)/\mathrm{Inn}_A(M).$$

If A is an algebra, the space $H^2(A, A)$ is related to *infinitesimal deformations* of A. This section is taken from notes from a lecture by Stroppel.

Let $k[\varepsilon]$ denote the algebra $k[x]/(x^2)$.

Definition 3.12. *If A is an algebra, an infinitesimal deformation of A is an associative $k[\varepsilon]$–bilinear product $*$ on the vector space $A \otimes_k k[\varepsilon]$ such that there is a bilinear $f : A \times A \to A$ with*

$$a * b = ab + f(a, b)\varepsilon$$

for all $a, b \in A$.

The fact that $*$ is associative gives a condition on f. For $a, b, c \in A$, we have

$$(a * b) * c = (ab + f(a, b)\varepsilon) * c = abc + f(ab, c)\varepsilon + f(a, b)c\varepsilon$$

and

$$a * (b * c) = a * (bc + f(b, c)\varepsilon) = abc + f(a, bc)\varepsilon + af(b, c)\varepsilon.$$

From this, we get

$$f(ab, c) + f(a, b)c = f(a, bc) + af(b, c)$$

for all $a, b, c \in A$. By the $k[\varepsilon]$–bilinearity, we see that $*$ is associative if and only if $f \in \ker(d : C^2(A, A) \to C^3(A, A))$.

Now suppose we have two infinitesimal deformations $*$ and $*'$ with $a * b = ab + f(a, b)\varepsilon$ and $a *' b = ab + g(a, b)\varepsilon$ for $a, b \in A$. Then $*$ and $*'$ are *equivalent* if there exists a $k[\varepsilon]$–bilinear isomorphism $\psi : (A \otimes k[\varepsilon], *) \to (A \otimes k[\varepsilon], *')$ with $\psi(a) = a + \gamma(a)\varepsilon$ for all $a \in A$, for some $\gamma : A \to A$. If $*$ and $*'$ are equivalent, we obtain

$$\psi(a * b) = \psi(ab + f(a, b)\varepsilon) = ab + \gamma(ab)\varepsilon + f(a, b)\varepsilon$$
$$\psi(a) *' \psi(b) = (a + \gamma(a)\varepsilon) *' (b + \gamma(b)\varepsilon) = ab + g(a, b)\varepsilon + \gamma(a)b\varepsilon + a\gamma(b)\varepsilon.$$

Thus, $*$ and $*'$ are equivalent if and only if $f - g = d\gamma$ for some $\gamma \in C^1(A, A)$. We conclude that $H^2(A, A)$ is in bijection with equivalence classes of infinitesimal deformations of A.

In general, $H^2(A,M)$ may be identified in a similar way with the vector space of k–algebra extensions of the form

$$0 \to M \to E \to A \to 0,$$

where M is an ideal in the algebra E with $M^2 = 0$, modulo the equivalence relation that two such extensions are equivalent if there is a commutative diagram of the form

$$\begin{tikzcd} & & E \arrow[dd, "\lambda"] \arrow[dr] & & \\ 0 \arrow[r] & M \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

with λ an algebra isomorphism. The isomorphism of $H^2(A,M)$ with this space may be realised by sending a cochain $f : A^{\otimes 2} \to M$ to the vector space $M \oplus A$ with the product $(m_1, a_1)(m_2, a_2) = (m_1a_2 + a_1m_2 + f(a_1, a_2), a_1a_2)$. The details will be given in Section 7 below.

Now let us consider $H_1(A,A)$. We calculate $A \otimes_{A^e} C_*^{bar}(A)$. It is an exercise to check that this comes out to be the following complex

$$\cdots A^{\otimes 3} \to A^{\otimes 2} \to A \to 0$$

with differential

$$d_H(a_0,\dots,a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0,\dots,a_ia_{i+1},\dots,a_n) + (-1)^n (a_na_0, a_1,\dots,a_{n-1}).$$

The space $H_1(A,A)$ is the first homology group of this complex, ie. $\ker(d_H : A^{\otimes 2} \to A)/\mathrm{Im}(d_H : A^{\otimes 3} \to A^{\otimes 2})$. If A happens to be commutative, this has a particularly nice description. In this case, $d_H(a_0,a_1) = a_0a_1 - a_1a_0 = 0$ and $d_H(a_0,a_1,a_2) = (a_0a_1,a_2) - (a_0,a_1a_2) + (a_2a_0,a_1)$. So

$$H_1(A,A) = \frac{A \otimes_k A}{\langle ab \otimes c - a \otimes bc + ca \otimes b : a,b,c \in A\rangle}.$$

If we write adb for $a \otimes b \in A \otimes_k A$, then we get

$$H_1(A,A) = \mathrm{span}_k\{adb : a,b \in A\}/\langle ad(bc) = abdc + acdb : a,b,c \in A\rangle,$$

which may also be described as the free A–module generated by the symbols db, $b \in A$, factored by the submodule generated by the following relations for $a,b \in A$, $\alpha \in k$:

$$d(\alpha) = 0, \quad d(a+b) = da + db, \quad d(ab) = adb + bda. \tag{1}$$

It may seem wrong that we are suddenly thinking of $H_1(A,A)$ as an A–module. However, it is easy to check from the definition that, for any algebra A, $H_i(A,A)$ always has a natural $Z(A)$–module structure. Hence if A is commutative, $H_1(A,A)$ is an A–module.

Definition 3.13. *Let A be a commutative k–algebra. The A–module $\Omega^1_{K\ddot{a}h}(A)$ freely generated by $\{db : b \in A\}$ with the relations* (1) *is called the* module of Kähler differentials *of A.*

Now we wish to compare Kähler differentials and global differential forms.

3.5. **Differential forms.** Let $A = k[X_1, X_2, \ldots, X_n]/I$ where I is a radical ideal. Then $A = k[X]$, the coordinate ring of the affine algebraic set in k^n given by the vanishing of the polynomials in I. For each $p \in X$, we have the maximal ideal $m_p \subset k[X]$ and the cotangent space $T^*_p X := m_p/m_p^2$ at p. By definition, the tangent space to X at p is the dual $T_p X := (m_p/m_p^2)^*$. We assume that X is irreducible.

Given $f \in k[X]$ and $x \in X$, we define $d_x f \in m_x/m_x^2$ by $d_x f := (f - f(x)) \mod m_x^2$. The *differential* of f is then defined to be the function $df : X \to \bigsqcup_{x \in X} T^*_p X$ which sends x to $d_x f$. Following the exposition in [Sha94, III, 5.1], we define $\Phi[X]$ to be the space of all functions $\phi : X \to \bigsqcup_{x \in X} T^*_p X$ such that $\phi(p) \in T^*_p X$ for all $p \in X$. These definitions also make sense for any open subset U of X.

We then make the following definition.

Definition 3.14. *A function $\phi \in \Phi[X]$ is called a* global differential form *on X if for every $p \in X$ there exists an open U containing p such that $\phi|_U$ belongs to the $k[U]$–submodule of $\Phi[U]$ generated by $\{df : f \in k[U]\}$.*

Differential forms are defined in the same way for every open subset of X, and it is clear from the nature of the definition that they form a sheaf Ω^1_X of $\mathcal{O}_X$–modules.

There is a natural map of $k[X]$–modules

$$\Omega^1_{K\ddot{a}h}(A) \to \Omega^1_X(X) = \Gamma(X, \Omega^1_X)$$

defined by $adf \mapsto (x \mapsto a(x)d_x f)$ for all $a, f \in A$ and all $x \in X$. This is a map of A–modules, and we now show that it is always surjective when X is irreducible.

Let $\phi \in \Omega^1_X(X)$. Then there exists an open cover $\{U_\alpha\}$ of X and $a_{\alpha,i}, f_{\alpha,i} \in k[U_\alpha]$ with $\phi|_{U_\alpha} = \sum_i a_{\alpha,i} df_{\alpha,i}$ for all α. By shrinking U_α if necessary, we may replace U_α by $U_{r_\alpha} = \{r_\alpha \neq 0\}$ for some $r_\alpha \in A$. By compactness of the Zariski topology, we may assume that the cover is finite. We have $a_{\alpha,i} = a'_{\alpha,i}/r^k_\alpha$ for some $k \in \mathbb{N}$ and $a'_{\alpha,i} \in A$, and $f_{\alpha,i} = f'_{\alpha,i}/r^\ell_\alpha$ for some $\ell \in \mathbb{N}$ and some $f'_{\alpha,i} \in A$. By the usual quotient rule for differentiation, which is easy to check from the definition of $d_p f$, we have $d_p(f/r) = r^{-2}(rd_p f - fd_p r)$. Using this, we see that there is some power g_α of r_α such that $g_\alpha \phi = \sum_i a''_{\alpha,i} df''_{\alpha,i}$ on U_α, with $a''_{\alpha,i}, f''_{\alpha,i} \in A$. Since the U_α were chosen to be a cover, the ideal (g_α) generated by the g_α must be the whole of A. Therefore, there exist $h_\alpha \in A$ with $\sum_\alpha h_\alpha g_\alpha = 1$. We then have

$$\phi = \sum_\alpha h_\alpha g_\alpha \phi = \sum_\alpha \sum_i h_\alpha a''_{\alpha,i} df''_{\alpha,i}$$

on $\bigcap_\alpha U_\alpha$. This is a nonempty dense open set because X is assumed to be irreducible. We will therefore be done, provided we can show that if $\sum v_i(x) d_x w_i = 0$ for all x in some dense open set, then $\sum v_i(x) d_x w_i = 0$ for all x. To see this, we note that for $w \in A$ and $x \in X$, $d_x w = 0$ if and only if $\sum_{i=1}^N \frac{\partial \widehat{w}}{\partial X_i}(x)(X_i - x_i) = 0$

where $\widehat{w}$ is an element of $k[X_1, \ldots, X_n]$ such that $\widehat{w} + I = w$. Therefore, the condition $\sum v_i(x) d_x w_i = 0$ is equivalent to the vanishing of a certain continuous function at x, and if this function vanishes on a dense U, then it vanishes on the whole of X.

We now wish to show that the natural map $\Omega^1_{K\ddot{a}h}(A) \to \Omega_X(X)$ need not be injective. We will use the following universal property of $\Omega^1_{K\ddot{a}h}(A)$.

Proposition 3.15. *If M is an A–module and $\delta : A \to M$ is a derivation then there exists a unique module map $\widehat{\delta} : \Omega^1_{K\ddot{a}h}(A) \to M$ such that $\widehat{\delta}(df) = \delta(f)$ for all $f \in A$.*

The proof of the proposition is an exercise. We say that $d : A \to \Omega^1_{K\ddot{a}h}(A)$ is the *universal derivation* from A to an A–module.

Corollary 3.16. *If $A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ then $\Omega^1_{K\ddot{a}h}(A)$ may be expressed as the following quotient of a free module:*

$$\Omega^1_{K\ddot{a}h}(A) \cong \frac{A \cdot dX_1 \oplus A \cdot dX_2 \oplus \cdots \oplus A \cdot dX_n}{A \cdot df_1 + \cdots A \cdot df_r}$$

where $df_i = \sum_{j=1}^N \frac{\partial f_i}{\partial X_j} dX_j$.

Proof. Show that the given module satisfies the universal property. □

Example 3.17. Let $A = k[x, y]/(y^2 - x^3) = k[C]$ where $C = V(y^2 - x^3) \subset \mathbb{C}^2$. Then $\Omega^1_{K\ddot{a}h}(A)$ is the free module on dx and dy, subject to the single relation $2ydy - 3x^2dx = 0$. Let $\omega = 2xdy - 3ydx$. We will show that $\omega \neq 0$. Indeed, if $\omega = 0$ then we get an equality $2xdy - 3ydx = a(x, y)(2ydy - 3x^2dx)$ in the free module on dx and dy, and this leads to a contradiction. Therefore, $\omega \neq 0$. However, $y\omega = 2xydy - 3y^2dx = 3x^3dx - 3x^3dx = 0$. Therefore, $\omega(x, y) = 0$ for all $(x, y) \in C$ with $y \neq 0$. Also, if $y = 0$ then $x = 0$ and so $\omega(0, 0) = 0$. Therefore, $\omega(x, y) = 0$ for all points (x, y) of C, and so ω is in the kernel of the map $\Omega^1_{K\ddot{a}h}(A) \to \Omega^1_C(C)$. Thus, this map need not be injective.

However, if X is a smooth variety then the map $\Omega^1_{K\ddot{a}h}(A) \to \Omega^1_X(X)$ is always injective. This is similar to the proof of surjectivity, using the fact that a system of local parameters may be found in a neighbourhood of each point. See [Sha94, Proposition 2, III 5.2] for a proof. Therefore,

Proposition 3.18. *If $A = k[X]$ is the coordinate ring of a smooth affine variety then $\Omega^1_{K\ddot{a}h}(A) \cong \Omega^1_X(X)$.*

For such A, we have shown that $H_1(A, A) = \Omega^1_{K\ddot{a}h}(A)$. Also, $H^1(A, A) = \mathrm{Der}(A)/\mathrm{Inn}(A) = \mathrm{Der}(A)$, which may be viewed as the global sections of the tangent sheaf TX. These facts generalise to the following important theorem.

Theorem 3.19 (Hochschild-Kostant-Rosenberg)**.** *If $A = k[X]$ is the coordinate ring of a smooth affine variety then*

$$HH_i(A) = \bigwedge_A^i \Omega^1_{K\ddot{a}h}(A) \cong \Omega^i_X(X)$$

$$H^i(A, A) = \bigwedge_A^i \mathrm{Der}(A) \cong \Gamma(X, \bigwedge^i TX)$$

for all $i \geq 0$.

We won't prove the Hochschild-Kostant-Rosenberg Theorem. A proof may be found in [Gin05, Section 9.2] or [Lod92, Theorem 3.4.4].

Thus, Hochschild homology may be viewed as a generalisation of differential forms to noncommutative algebras. However, there is no obvious way to define "d" on Hochschild homology. Therefore, if we wish to define a calculus of differential forms on noncommutative algebras, we need to find another approach. This is the goal of the first part of [CQ95a], which we explain in the next section.

Exercises 3.20. Exercises on Hochschild (co)homology and differential forms.

(1) For any A, show that $HH_i(A)$ is naturally a $Z(A)$–module.
(2) Show that $A \mapsto HH_i(A)$ is a functor from the category of algebras to the category of vector spaces, but $A \mapsto H^i(A, A)$ is not.
(3) Use Morita invariance of HH_0 to show that $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$.
(4) Show that if $HH_0(A) = 0$ then A has no nonzero finite-dimensional representations.
(5) Show that the complex $A \otimes_{A^e} C^{bar}_*(A)$ is isomorphic to the complex whose n^{th} term is $A^{\otimes(n+1)}$ and whose differential is given by the formula

$$d_H(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}).$$

(6) Compute $\Omega^1_X(X)$ where $X = \mathbb{P}^1$.
(7) Prove Proposition 3.15.
(8) Prove Proposition 3.16.

4. Noncommutative forms

Let A be a commutative k–algebra. Recall from Proposition 3.15 that $\Omega^1_{K\ddot{a}h}(A)$ may be characterised by the property that $d : A \to \Omega^1_{K\ddot{a}h}(A)$ is a derivation, and for every derivation $\delta : A \to M$, there exists a unique $\widehat{\delta} : \Omega^1_{K\ddot{a}h}(A) \to M$ with $\widehat{\delta} d = \delta$. This can be expressed in the following way

$$\mathrm{Der}_A(M) \cong \mathrm{Hom}_{A-\mathrm{Mod}}(\Omega^1_{K\ddot{a}h}(A), M)$$

for every A–module M. This isomorphism is natural in M, so can be viewed as an isomorphism of functors $A - \mathrm{Mod} \to \mathrm{Vect}_k$

$$\mathrm{Der}_A(-) \cong \mathrm{Hom}_{A-\mathrm{Mod}}(\Omega^1_{K\ddot{a}h}(A), -).$$

In the language of category theory, the object $\Omega^1_{K\ddot{a}h}(A)$ is said to *represent* the functor $M \mapsto \mathrm{Der}_A(M)$.

For a noncommutative algebra A and a left A–module M, $\mathrm{Der}_A(M)$ doesn't make sense. However, if M is a bimodule, we may define

$$\mathrm{Der}_A(M) := \{d : A \to M : d(ab) = adb + (da)b \quad \forall a, b \in A\}.$$

We may then ask the following question:

Does there exist an A–bimodule Ω^1_A which represents the functor $M \to \mathrm{Der}_A(M)$ form $A - \mathrm{Bimod}$ to Vect_k?

(Notice that we don't yet know the answer to this, even if A is commutative, because even for a commutative ring, bimodules and modules are quite different notions. For example, if $A = k[X]$, an A–bimodule is a $A \otimes A^{op} = k[X, Y]$–module.)

The answer to the question is yes. We may define the bimodule Ω^1_A as follows. Let A be an algebra and $\overline{A} = A/k \cdot 1_A$, a vector space. Define

$$\Omega^1_A = A \otimes_k \overline{A}.$$

Equip this with the natural left A–module structure, and define the right A–module structure by

$$(a \otimes \overline{b})c = a \otimes \overline{bc} - ab \otimes \overline{c}$$

for all $a, b, c \in A$, where $\overline{(-)}$ denotes the quotient map. It can be quickly checked that this right action is well-defined, and that it really makes Ω^1_A into a right A–module and a bimodule.

Proposition 4.1. *There is an isomorphism of functors*

$$\mathrm{Der}_A(-) \cong \mathrm{Hom}_{A-\mathrm{Bimod}}(\Omega^1_A, -).$$

Proof. Given a bimodule M, the map $\mathrm{Der}_A(M) \cong \mathrm{Hom}_{A-\mathrm{Bimod}}(\Omega^1_A, M)$ is defined by $\delta \mapsto (a \otimes \overline{b} \mapsto a\delta(b))$, while the map $\mathrm{Hom}_{A-\mathrm{Bimod}}(\Omega^1_A, M) \to \mathrm{Der}_A(M)$ is defined by $\psi \mapsto (a \mapsto \psi(1 \otimes a))$. Everything has been set up so that these maps are well-defined and provide a natural isomorphism. □

To define forms of higher degree, we need the notion of a differential graded algebra.

4.1. **Differential graded algebras.**

Definition 4.2. *A* differential graded algebra*, or dga for short, is a graded k–algebra*

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

together with a linear map $d : A \to A$ such that $d(A_i) \subset A_{i+1}$ for all i, and $d^2 = 0$, satisfying the Leibniz rule*:*

$$d(ab) = da \cdot b + (-1)^i adb$$

for all $a \in A_i$, $b \in A_j$, $i, j \in \mathbb{Z}$.

For a homogeneous element $a \in A_i$, we often write $i = |a|$ as shorthand for the degree of a.

Examples 4.3. Here are some examples of dgas.

(1) Any algebra A is a dga if we take $A_0 = A$ and $A_i = 0$ for $i \neq 0$.

(2) A more interesting example: if X is an affine variety then $\bigoplus_{i=0}^{\infty} \Omega^i_X(X)$ is a dga with product given by the wedge product of forms, and the usual de Rham differential d.

(3) If A is an algebra and I is a 2–sided ideal of A, we can define a dga $A_{-1} \oplus A_0$ with $A_{-1} = I$, $A_0 = A$, and the differential given by the inclusion map $I \hookrightarrow A$. The Leibniz rule has to be checked.

(4) If A is an algebra and $\delta : A \to M$ is a derivation, with M a bimodule over A, then there is a dga $A_0 \oplus A_1$ with $A_0 = A$, $A_1 = M$ and the differential given by $\delta : A \to M$.

(5) If $(V_\bullet, d)$ is any complex of vector spaces, a linear map $f : V \to V$ is said to have degree n, $n \in \mathbb{Z}$, if $f(V_i) \subset V_{i+n}$ for all i. We can construct a dga of linear maps $V_\bullet \to V_\bullet$ via the following lemma.

Lemma 4.4. *Let $(V_\bullet, d)$ be a complex of vector spaces, where d has degree $+1$. For each $n \in \mathbb{Z}$, let E_n be the space of all linear maps $f : V \to V$ of degree n. Then $\bigoplus_{n \in \mathbb{Z}} E_n$ is a dga with multiplication given by composition of functions and the differential defined by*

$$(\partial\psi)(w) = d(\psi(w)) - (-1)^n \psi(dw)$$

for $\psi \in E_n$ and $w \in V_i$, $i \in \mathbb{Z}$.

Proof. To show that $\partial^2 = 0$, for $\psi \in E_n$ we have

$$\partial^2\psi = \partial(\partial\psi) = \partial(d\psi - (-1)^n \psi d) = d(d\psi - (-1)^n \psi d) - (-1)^{n+1}(d\psi - (-1)^n \psi d)d = 0,$$

where we used the fact that $\partial\psi$ has degree $n + 1$.

To show that the Leibniz rule holds, we calculate for any $\phi \in E_{|\phi|}$ and $\psi \in E_{|\psi|}$,

$$\partial\phi \cdot \psi + (-1)^{|\phi|}\phi\partial\psi = (d\phi - (-1)^{|\phi|}\phi d)\psi + (-1)^{|\phi|}\phi(d\psi - (-1)^{|\psi|}\psi d) = d\phi\psi - (-1)^{|\phi|+|\psi|}\phi\psi d = \partial(\phi\psi)$$

because $|\phi\psi| = |\phi| + |\psi|$. □

4.2. **The dg algebra of noncommutative forms.** Let A be a k–algebra. In [CQ95a, Section 1], it is shown that there is a *universal* dga ΩA with $(\Omega A)_0 = A$. We can think of ΩA as a kind of dg-enveloping algebra of A, or the dg-algebra freely generated by A. This dg-algebra will be constructed as the dg-algebra of *noncommutative forms* on A. The notion of noncommutative forms is originally due to Connes. They are defined as follows.

Definition 4.5. *Given an algebra A, let $\overline{A} = A/k1_A$ as before. Define*

$$\Omega^n(A) = A \otimes_k \overline{A}^{\otimes n}$$

for $n \geq 0$, and set $\Omega^n(A) = 0$ for $n < 0$. Call $\Omega^n(A)$ the space of differential n–forms on A. *Define*

$$\Omega A = \bigoplus_{n \in \mathbb{Z}} \Omega^n(A)$$

and define $d : \Omega^n(A) \to \Omega^{n+1}(A)$ by

$$d(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n)$$

for $a_i \in A$.

Note that in the definition of d, we write $(a_0, a_1, \ldots, a_n)$ for $a_0 \otimes a_1 \otimes \cdots \otimes a_n$. Also, $a_1, \ldots, a_n$ are really elements of $\overline{A}$, but it is easy to check that d given by the above formula is well-defined.

To give ΩA the structure of a dg-algebra, we will use the following lemma.

Lemma 4.6. *In any dg algebra B, the following identities hold:*

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n \tag{2}$$

$$(a_0 da_1 \cdots da_n)(a_{n+1} da_{n+2} \cdots da_k) = (-1)^n a_0 a_1 da_2 \cdots da_k + \sum_{i=0}^{n-1} (-1)^{n-i} a_0 da_1 \cdots d(a_i a_{i+1}) \cdots da_n \tag{3}$$

where $a_0, \ldots, a_k \in B$.

Proof. Statement (2) is immediate from the Leibniz rule. Statement (3) follows from expanding $d(a_i a_{i+1})$ via the Leibniz rule and cancelling terms pairwise. $\square$

Proposition 4.7. [CQ95a, Proposition 1.1] *The complex $(\Omega A, d)$ is a dg algebra under the product*

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_k) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_i a_{i+1}, \ldots, a_n). \tag{4}$$

There is a natural map of algebras $i : A \to \Omega^0(A)$. Furthermore, if $\Gamma = \bigoplus_i \Gamma_i$ is a dg algebra, and $\psi : A \to \Gamma_0$ is an algebra map, then there exists a unique morphism $\psi' : \Omega A \to \Gamma$ of dga's such that $\psi' \circ i = \psi$.

Proof. The following argument is due to Cuntz and Quillen. The idea is to construct the multiplication on ΩA by realising ΩA as a subalgebra of a dg algebra.

Let E be the dga of linear maps from the complex ΩA to itself. For $a \in A$, define $\ell_a \in E$ by $\ell_a(a_0, \ldots, a_n) = (aa_0, a_1, \ldots, a_n)$. Then define a graded linear map $\lambda : \Omega A \to E$ by

$$\lambda(a_0, \ldots, a_n) = \ell_{a_0} \partial(\ell_{a_1}) \cdots \partial(\ell_{a_n}).$$

To check that λ is well-defined, we need to check that $\partial(\ell_a)$ depends only on the class of $a \in \overline{A}$. This essentially reduces to checking that $\partial(1_A)(a_0, \ldots, a_n) = 0$ for all $(a_0, \ldots, a_n)$, which is straightforward.

Now we show that $\mathrm{Im}(\lambda)$ is a dg-subalgebra of E. From (2), we see that $\mathrm{Im}(\lambda)$ is closed under d. From (3), we see that $\mathrm{Im}(\lambda)$ is closed under multiplication. It follows that $\mathrm{Im}(\lambda)$ is a dg-subalgebra of E.

Now define a linear map $\mu : E \to \Omega A$ by $\mu(\psi) = \psi(1_A)$ for $\psi \in E$. We verify that $\mu\lambda$ is the identity on ΩA. For this, we need to show that $\ell_{a_0}\partial(\ell_{a_1}) \cdots \partial(\ell_{a_n})(1_A) = (a_0, a_1, \dots, a_n)$ for all $a_0, \dots, a_n \in A$. Now,

$$\partial(\ell_{a_n})(1_A) = d\ell_{a_n}(1_A) - (-1)^{|\ell_{a_n}|}\ell_{a_n} d(1_A) = (1, a_n)$$

and

$$\partial(\ell_{a_{n-1}})(1, a_n) = (1, a_{n-1}, a_n) - \ell_{a_{n-1}} d(da_n) = (1, a_{n-1}, a_n),$$

and continuing inductively, we obtain

$$\ell_{a_0}\partial(\ell_{a_1}) \cdots \partial(\ell_{a_n})(1_A) = (a_0, a_1, \dots, a_n)$$

as desired. This shows that $\mu\lambda = \mathrm{id}_{\Omega A}$. Therefore, $\lambda : \Omega A \to \mathrm{Im}(\lambda)$ is injective, hence it is an isomorphism of graded vector spaces. Therefore, we may use the product on $\mathrm{Im}(\lambda)$ to make ΩA into a dg-algebra. Formula (3) applied in the dg-algebra $\mathrm{Im}(\lambda)$ then immediately implies that we get the product (4). We also need to check that the differential induced on ΩA via the isomorphism λ coincides with the original differential d. To see this, we should check that

$$\mu\partial\lambda(a_0, \dots, a_n) = d(a_0, \dots, a_n)$$

for all $a_0, \dots, a_n \in A$. This is simple to check using the definitions of μ and λ.

We have shown that $(\Omega A, d)$ becomes a dg-algebra under the product (4). It remains to check the universal property. For this, suppose Γ is a dg-algebra and $\psi : A \to \Gamma_0$ is a map of algebras. Define $\psi' : \Omega A \to \Gamma$ by

$$\psi'(a_0, a_1, \dots, a_n) = \psi(a_0) d\psi(a_1) \cdots d\psi(a_n). \tag{5}$$

To see that ψ' is the unique dg-algebra homomorphism which extends ψ, we note that

$$(a_0, \dots, a_n) = a_0 da_1 da_2 \cdots da_n$$

for all $a_0, \dots, a_n \in A$. This follows from the definition of the product on ΩA. Therefore, if ψ' is a dg-algebra map which agrees with ψ in degree 0, it must be given by (5). It remains to check that ψ' preserves d and multiplication. This can be checked using identities (2) and (3) in the dg-algebra Γ. $\square$

4.3. **Relative forms.** Now we study the relative version of differential forms, following [CQ95a, Section 3].

Let S and A be k–algebras. We say that A is an S-algebra if there exists an algebra morphism $S \to A$. If A is an S–algebra, we will identify S with its image in A, and we write A/S for the quotient vector space, which is also an S–bimodule.

Definition 4.8. *Let A be an S–algebra. The space of* differential n–forms on A relative to S *is*

$$\Omega^n_S(A) = A \otimes_S (A/S) \otimes_S \cdots \otimes_S (A/S)$$

where there are n copies of A/S in the tensor product.

Define a graded vector space $\Omega_S A$ by

$$\Omega_S A := \bigoplus_{i=0}^{\infty} \Omega_S^n(A).$$

In order to make $\Omega_S A$ into a dga, we realize it as a quotient of ΩA. This requires the notions of dg ideal and quotient dg-algebra, which generalize the notions of ideal and quotient for ordinary algebras.

Definition 4.9. *Let $E = \bigoplus_{i\in\mathbb{Z}} E_i$ be a dg-algebra with differential d. A graded subspace $I = \bigoplus_{i\in\mathbb{Z}} I_i \subset E$ is called a* dg ideal *if it is an ideal, and $d(I) \subset I$. If $I \subset E$ is a dg-ideal, the* quotient *dg-algebra is the vector space $\bigoplus_{i\in\mathbb{Z}}(E_i/I_i)$ equipped with the differential induced from d and the multiplication being the multiplication on E/I.*

Proposition 4.10. *For each $n \geq 0$, $\Omega_S^n(A)$ is the quotient space of $\Omega^n(A)$ by the subspace spanned by the elements*

$$(a_0, \dots, a_{i-1}s, a_i, \dots, a_n) - (a_0, \dots, a_{i-1}, sa_i, \dots, a_n),$$

and

$$(a_0, \dots, a_{i-1}, s, a_{i+1}, \dots, a_n)$$

for $a_0 \in A$, $a_i \in \overline{A}$ $(1 \leq i \leq n)$, $s \in S$.

Proof. Define a linear map $\Omega^n(A) \to \Omega_S^n(A)$ via $(a_0, a_1, \dots, a_n) \mapsto (a_0, a_1 + S, \dots, a_n + S)$. The definition of tensor product shows that the kernel is spanned by the given relations. □

We see from the definition of d that $\Omega_S A$ is a quotient of ΩA by a dg ideal, and hence it becomes a dg-algebra. In fact, it is a dg S–algebra, that is, there is a natural map $S \to \Omega_S(A)$ which is a map of dg-algebras, which amounts to saying that the image of S is contained in the kernel of d.

Proposition 4.11. [CQ95a, Proposition 2.1] *The dg-algebra $\Omega_S A$ is the universal dg S–algebra generated by A, in the sense that if Γ is a dg S–algebra and $\psi : A \to \Gamma$ is a map of S–algebras, then there exists a unique map of dg S–algebras $\psi' : \Omega_S A \to \Gamma$ satisfying $\psi'|_{\Omega_S^0(A)} = \psi$.*

Proof. An exercise, using Proposition 4.7. □

4.4. **Other descriptions of $\Omega_S A$.** Now we give some other descriptions of $\Omega_S A$, which can be very useful in calculations. Everything we say for $\Omega_S A$ also holds for ΩA, since $\Omega A = \Omega_S A$ with $S = k$.

Definition 4.12. *Let A be an algebra and M an A–bimodule. The* tensor algebra $T_A(M)$ *of M over A is the graded algebra*

$$T_A(M) = \bigoplus_{i=0}^{\infty} T_A(M)_i = \bigoplus_{i=0}^{\infty} \underbrace{M \otimes_A M \otimes_A \cdots \otimes_A M}_{i \text{ copies}}$$

where $T_A(M)_0 := A$, with multiplication given by concatenation.

This algebra has the following universal property. If A is an algebra and R is an A–algebra and $\psi : M \to R$ is a map of A–bimodules, then there exists a unique A–algebra map $\psi' : T_A(M) \to R$ such that the restriction of ψ' to $M = T_A(M)_1$ is ψ.

The map ψ' in the universal property is given explicitly by $\psi'(a_0 \otimes a_1 \cdots \otimes a_n) = \psi(a_0)\psi(a_1)\cdots\psi(a_n)$.

Proposition 4.13. [CQ95a, Proposition 2.3] *Let A be an S–algebra. Then*

$$\Omega_S A \cong T_A(\Omega^1_S(A))$$

as graded S–algebras.

Proof. The universal property of the tensor algebra given a linear map

$$\lambda_n : T_A(\Omega^1_S(A))_n \to \Omega^n_S(A)$$

for every n, defined by $\omega_1 \otimes \cdots \otimes \omega_n \mapsto \omega_1\omega_2\cdots\omega_n$. It suffices to show that λ_n is an isomorphism for every n. In degrees $n = 0, 1$, it is the identity. We show by induction that it is an isomorphism in all degrees. Suppose λ_n is an isomorphism. Then

$$T_A(\Omega^1_S(A))_{n+1} = T_A(\Omega^1_S(A))_n \otimes_A \Omega^1_S(A) \cong \Omega^n_S(A) \otimes_A \Omega^1_S(A)$$

where the last isomorphism is via $\lambda_n \otimes \mathrm{id}$. But

$$\Omega^n_S(A) \otimes_A \Omega^1_S(A) = A \otimes_S (A/S) \otimes_S \cdots \otimes_S (A/S) \otimes_A (A \otimes_S (A/S))$$

which is isomorphic to

$$A \otimes_S (A/S) \otimes_S \cdots \otimes_S (A/S) \otimes_S (A/S) = \Omega^{n+1}_S(A)$$

via $\omega \otimes b_0 db_1 \mapsto \omega b_0 db_1$. Therefore, λ_{n+1} is also an isomorphism, as required. □

Another way to describe the differential 1–forms is as the kernel of the multiplication map.

Proposition 4.14. [CQ95a, Proposition 2.5] *Let A be an S–algebra. There exists a short exact sequence of A–bimodules*

$$0 \longrightarrow \Omega^1_S(A) \xrightarrow{j} A \otimes_S A \xrightarrow{m} A \longrightarrow 0$$

where $j(a, b) = a \otimes b - ab \otimes 1$ and $m(x \otimes y) = xy$ for all $a, x, y \in A$ and all $b \in \overline{A}$. (In the definition of j, we use the notation (a, b) for $a \otimes b \in A \otimes_k \overline{A}$ as before.)

Proof. By definition, m is a bimodule map. It is an exercise to check that j is a well-defined bimodule map. To see that j is injective, define $k : A \otimes_S A \to A \otimes_S (A/S)$ to be the projection. Note that k is a linear map with no extra structure. We have $kj = \mathrm{id}$ on $\Omega^1_S(A)$, and so j is injective. Clearly, $\mathrm{im}(j) \subset \ker(m)$. Now suppose $\sum_i a_i \otimes b_i \in \ker(m)$. Then $\sum_i a_i b_i = 0$ so $\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i - \sum_i a_i b_i \otimes 1 = \sum_i (a_i \otimes b_i - a_i b_i \otimes 1) \in \mathrm{im}(j)$, and we are done. □

As a special case of Proposition 4.14, we obtain

$$\Omega^1(A) = \ker(A \otimes_k A \to A),$$

which is often used as the definition of $\Omega^1(A)$ in the literature.

5. Applications of noncommutative forms

In [CQ95a], Cuntz and Quillen go on to use noncommutative forms to construct solutions to a variety of universal problems. Inspired by this, we can define generalized (or "curved") representations of algebras.

5.1. **Generalized representations.**

The notion of an algebra homomorphism can be weakened to that of *based linear map.*

Definition 5.1. *If A and B are algebras, a* based linear map *$\varrho : A \to B$ is a linear map $\varrho : A \to B$ such that $\varrho(1_A) = 1_B$.*

If A and B are S–algebras, a based linear S–map *$\varrho : A \to B$ is an S–bimodule map $\varrho : A \to B$ such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "\varrho"] & & B \\ & S \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

The curvature *of a based linear map $\varrho : A \to B$ is the linear map $\omega : A \otimes A \to B$ defined by*

$$\omega(a \otimes b) = \varrho(ab) - \varrho(a)\varrho(b).$$

A generalized representation *of a k–algebra A is a based linear map $\varrho : A \to \mathrm{End}_k(V)$ where V is a k–vector space.*

Based linear S maps correspond to representations of a certain algebra constructed from A. If A is an S–algebra, then A is an S–bimodule. Therefore, we may form the tensor algebra $T_S(A) = \bigoplus_{i=0}^{\infty} T_S(A)_i$. For $s \in S$, we write s_0 for the copy of s in $T_S(A)_0 = S$ and s_1 for the copy of s in $T_S(A)_1 = A$.

Definition 5.2. *Let A be an S–algebra. Define*

$$R_S A = T_S(A)/I$$

where I is the two-sided ideal generated by the elements $s_0 - s_1$ for $s \in S$.

Proposition 5.3. *Let A and R be S–algebras. Then based linear maps $A \to R$ correspond to S–algebra maps $R_S A \to R$.*

Proof. Given a based linear map $\varrho : A \to R$, we define an S–algebra map $\widehat{\varrho}$ via $\widehat{\varrho}(a_1 \otimes \cdots \otimes a_n) = \varrho(a_1) \cdots \varrho(a_n)$ in degree n. It is easy to see that this gives a bijective correspondence, as required. □

An important special case of Proposition 5.3 is when we take $\varrho : A \to A$ to be the identity. In this case, there is an induced S–algebra map $R_S A \to A$ and we define IA to be the kernel of this map, a two-sided ideal in $R_S A$. Note that $R_S A/IA \cong A$.

Here is another description of $R_S A$.

Lemma 5.4. *If S is a semisimple algebra then $R_S A \cong T_S(A/S)$. In particular, if $S = k$ then $R_k A$ is free.*

Proof. A based linear S–map $A \to R$ is the same thing as as S–algebra map $S \to R$ together with a map of A–bimodules $A/S \to R$. But, by the universal property of $T_S(A/S)$, this is the same as an S–algebra map $T_S(A/S) \to R$. $\square$

Example 5.5. Recall that a *quiver* is a finite directed graph. This can be viewed as a 4–tuple (Q_0, Q_1, h, t) where Q_0 is a finite set called the set of vertices of Q, Q_1 is a finite set called the set of arrows of Q, and $h, t : Q_1 \to Q_0$ are functions assigning to an arrow its head and tail respectively. Note that loops and parallel edges are allowed. Let $n = |Q_0|$ be the number of vertices.

Let S be the semisimple commutative algebra $k \times k \cdots \times k$ where there are n copies of k. Write e_i for the element $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i^{th} position. Then we may write $S = \bigoplus_{i=1}^n ke_i$. Any bimodule M over S satisfies $M = \bigoplus_{i,j} e_i M e_j$, and so is just given by a collection of vector spaces indexed by $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$.

We associate such a bimodule M_Q to Q by setting $e_i M_Q e_j$ to be a vector space with a basis given by the arrows a with $h(a) = j$ and $t(a) = i$. We may then define the *path algebra* of Q to be the algebra $kQ := T_S(M_Q)$. This is a k–algebra with a basis given by the paths in Q, that is, sequences $a_1 a_2 \cdots a_n$ of edges such that $h(a_i) = t(a_{i-1})$ for all i. The *trivial paths* e_i also count as paths in Q (they span the degree 0 component of kQ).

We may now ask what a generalized representation of kQ is, relative to S. If we let $A = kQ$, then A/S is a vector space spanned by all nontrivial paths in Q. Thus, for each i, j, $e_j(A/S)e_i$ is the number of paths from i to j in Q. Therefore, $T_S(A/S)$ is the path algebra of the quiver $\overline{Q}$, where $\overline{Q}$ has an arrow $i \to j$ for each nontrivial *path* $i \to j$ in Q.

For example, if Q is the quiver

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

then $\overline{Q}$ is the following quiver.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Now we explain the connection between generalized representations and noncommutative forms.

Definition 5.6. *Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a dga. The* Fedosov product *on A is defined by*

$$x \circ y = xy - (-1)^{|x|} dx dy$$

for homogeneous elements $x, y \in A$.

It is an exercise, directly applying the Leibniz rule, to show that the Fedosov product is associative. Under the Fedosov product, $(A, \circ)$ is an algebra, but it is no longer a graded algebra.

Proposition 5.7. [CQ95a, Proposition 1.2] *If A is an A–algebra then $R_S A$ is isomorphic to the S–algebra $\Omega_S^{ev}(A)$ of even-dimensional forms under the Fedosov product. Furthermore, under this isomorphism, the ideal IA^n corresponds to $\bigoplus_{k>n} \Omega_S^{2k}(A)$.*

Proof. The following argument is given by Cuntz and Quillen in the case $S = k$; it goes through without change in the case of a general S.

Define a based linear S–map $\rho : A \to \Omega_S^{ev}(A)$ by $\rho(a) = a \in \Omega_S^0(A)$. The curvature of ρ is $\omega_\rho(a, b) = ab - a \circ b = ab - (ab - (-1)^0 daab) = dadb$.

Corresponding to ρ, we have a morphism of S–algebras $\Psi : R_S A \to \Omega_S^{ev}(A)$. By definition, this Ψ satisfies

$$\Psi(a_0 \otimes (a_1 a_2 - a_1 \otimes a_2) \otimes (a_3 a_4 - a_3 \otimes a_4) \cdots) = a_0 da_1 da_2 \cdots$$

for all $a_0, a_1, \ldots \in A$, and so Ψ is a surjective map of S–algebras. We now show that Ψ is an isomorphism.

Define a *linear* map $\Phi_{2k} : \Omega^{2k}(A) \to (T_S(A/S))_{2k}$ by

$$\Phi_{2k}(a_0, a_1, \ldots, a_{2k}) = a_0 \otimes (a_1 a_2 - a_1 \otimes a_2) \otimes \cdots \otimes (a_{2k-1} a_{2k} - a_{2k-1} \otimes a_{2k}).$$

Then Φ_{2k} is well-defined as a map from $A \otimes_k \overline{A}^{\otimes(2k)}$, and it respects the defining relations of $\Omega_S^{2k}(A)$. Therefore, it gives a well-defined linear map $\Phi_{2k} : \Omega_S^{2k}(A) \to R_S A$. We can put all the Φ_{2k} together to give a linear map $\Phi : \Omega_S^{ev}(A) \to R_S A$. We show that Φ is surjective. Indeed, the image of Φ is a left ideal in $R_S A$ and contains $1_{R_S A}$, which is enough to show that Φ is surjective. Furthermore, $\Psi\Phi$ is the identity, because $(a_0, a_1, \ldots, a_n) = a_0 da_1 \cdots da_n$ in ΩA. Therefore, Φ is also injective, so it is a bijection and is inverse to the S–algebra map Ψ. It follows that Φ is also an S–algebra map and so Ψ is an isomorphism.

It remains to show the statement about IA, which recall is defined as the kernel of the natural map $R_S A \to A$ induced by the identity map $A \to A$. In particular, $a_1 a_2 - a_1 \otimes a_2 \in IA$ for all $a_1, a_2 \in A$. Thus, $\Omega_S^2(A) \subset \Psi(IA)$ and hence $\bigoplus_{k \geq n} \Omega_S^{2k}(A) \subset \Psi(IA^n)$. For the reverse inclusion, note that the following diagram commutes, where the map $\Omega_A^{ev}(A) \to A$ is projection onto the 0–degree component.

$$\begin{tikzcd}
R_S A \arrow[dd, "\Psi"'] \arrow[dr] & \\
& A \\
\Omega_S A \arrow[ur] &
\end{tikzcd}$$

Since Ψ is an isomorphism, it follows that $\Psi(IA) = \bigoplus_{k \geq 1} \Omega_S^{2k}(A)$ and therefore $\bigoplus_{k \geq n} \Omega_S^{2k}(A) \supset \Psi(IA^n)$. $\square$

Corollary 5.8. *The ideal $IA^n \subset R_S A$ is generated by the elements*

$$(a_1a_2 - a_1 \otimes a_2) \otimes (a_3a_4 - a_3 \otimes a_4) \otimes \cdots \otimes (a_{2n-1}a_{2n} - a_{2n-1} \otimes a_{2n})$$

for $a_1, \dots, a_{2n} \in A$.

5.2. **Nilpotent generalized representations.** It is interesting to see what kinds of objects we get by imposing conditions on the curvature of a generalized representation $\varrho : A \to \mathrm{End}(V)$ which are weaker than $\omega = 0$. One obvious one is $\omega^{\otimes n} = 0$. In this case, $\varrho : A \to \mathrm{End}(V)$ is said to be *nilpotent of degree n*.

Corollary 5.8 implies that ϱ is nilpotent of degree n if and only if

$$(\varrho(a_1a_2) - \varrho(a_1)\varrho(a_2))(\varrho(a_3a_4) - \varrho(a_3)\varrho(a_4)) \cdots (\varrho(a_{2n-1}a_{2n}) - \varrho(a_{2n-1})\varrho(a_{2n})) = 0$$

for all $a_i \in A$. Nilpotent generalized representations of degree n correspond to representations of $R_S A/IA^n$. For simplicity, we take $S = k$. What can be said about nilpotent generalized representations?

If V is a nilpotent generalized representation of A of degree n, then we can filter V as

$$V \supset IA \cdot V \supset IA^2 \cdot V \supset \cdots \supset IA^n \cdot V = 0.$$

The associated graded space $\mathrm{gr}(V) = \bigoplus_i IA^i \cdot V/IA^{i+1} \cdot V$ is a representation of $RA/IA = A$. Therefore, we see that an algebra A has a nilpotent generalized representation of dimension n if and only if it has an ordinary representation of dimension n. Furthermore, every nilpotent generalized representation V has a basis with respect to which the $\varrho(a)$ have the form

$$\varrho(a) = \begin{pmatrix} \rho_1(a) & f_{12}(a) & \cdots & f_{1n}(a) \\ 0 & \rho_2(a) & \cdots & f_{2n}(a) \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \rho_n(a) \end{pmatrix}$$

where the ρ_i are representations $A \to V_i$, and the f_{ij} are linear maps $A \to \mathrm{Hom}(V_i, V_j)$ which satisfy $f_{ij}(1_A) = 0$. This can be refined to get a description of the category $RA/IA^n - \mathrm{Mod}$.

5.3. **Superalgebras.** Another universal problem whose solution is contained in ΩA is the following.

Definition 5.9. *A* superalgebra *is an algebra A which is $\mathbb{Z}/2$–graded. That is, as a vector space, $A = A_+ \oplus A_-$ where A_+ is a subalgebra, and $A_+A_- \subset A_- \supset A_-A_+$, while $A_-A_- \subset A_+$.*

Equivalently, a superalgebra is a pair (A, γ) where A is an algebra and $\gamma : A \to A$ is an automorphism with $\gamma^2 = 1$.

Given an algebra A, there exists a superalgebra QA with the following universal property: there is an algebra map $A \to QA$, and if U is any superalgebra with an algebra map $A \to U$, then there exists a unique map $QA \to U$ of superalgebras which makes the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & QA \arrow[dl] \\ U & \end{tikzcd}$$

commute.

The superalgebra QA may be constructed as the coproduct $A * A$ in the category of k–algebras. Recall that the category of k–algebras has coproducts. Indeed, given k–algebras A and B with presentations $A = k\langle S\rangle/I_A$, $B = k\langle T\rangle/I_B$, where $k\langle X\rangle$ denotes the free algebra on a set X, we may define

$$A * B = k\langle S \sqcup T\rangle/I \tag{6}$$

where I is the two-sided ideal generated by I_A and I_B. It is an exercise to check that this is indeed a coproduct of A and B.

The algebra $A * A$ has a natural automorphism γ of order 2. This can be seen from the categorical definition. Explicitly, it is given by interchanging the generators of the two copies of A in the presentation (6).

Theorem 5.10. [CQ95a, Proposition 1.3] *Let A be a k–algebra. There is an isomorphism of superalgebras between QA and $(\Omega A, \circ)$ where $\circ$ denotes the Fedosov product.*

The proof of Proposition 5.10 is similar to the proof of Proposition 5.7, and can be found in the first section of [CQ95a] (where more is also proved). Cuntz and Quillen then go on to give a little-known but very interesting description of the free product $A * B$ of any two algebras in terms of ΩA and ΩB (see [CQ95a, Proposition 1.4]). It is remarkable that the solutions of so many universal problems should be expressible in terms of differential forms.

5.4. **Differential forms and Hochschild cohomology.** Recall the *reduced bar complex* of an algebra A. This is a projective resolution of A in the category of A–bimodules, and it has the form

$$\cdots \to A \otimes_k \overline{A}^{\otimes 2} \otimes A \to A \otimes \overline{A} \otimes A \to A \to 0$$

with the differential b' given by

$$b'(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

We can identify $A \otimes \overline{A}^{\otimes n}$ with $\Omega^n(A)$, and this enables us to rewrite b' in terms of multiplication of forms. Recall that $(a_0, \ldots, a_n) = a_0 da_1 \cdots da_n$ in $\Omega^n(A)$. We have

$$b'(a_0 da_1 \cdots da_n \otimes a_{n+1}) = \sum_{i=1}^{n-1} (-1)^i a_0 da_1 \cdots d(a_i a_{i+1}) \cdots da_n \otimes a_{n+1}$$
$$+ (-1)^n a_0 da_1 \cdots da_{n-1} \otimes a_n a_{n+1} + a_0 a_1 da_2 \cdots da_n \otimes a_{n+1}.$$

Now recall the rule for multiplication of forms:

$$(a_0 da_1 \cdots da_{n-1}) a_n = \sum_{i=0}^{n-1} (-1)^{n-1-i} a_0 \cdots d(a_i a_{i+1}) \cdots da_n.$$

We thus obtain

$$b'(a_0 da_1 \cdots da_n \otimes da_{n+1}) = (-1)^{n-1} (a_0 da_1 \cdots da_{n-1}) a_n \otimes a_{n+1} + (-1)^n (a_0 da_1 \cdots da_{n-1} \otimes a_n a_{n+1}$$

which may be written in general as

$$b'(\omega da \otimes a') = (-1)^{|\omega|} (\omega a \otimes a' - \omega \otimes a a'), \tag{7}$$

which is Equation 24 in [CQ95a]. We will use (7) to study algebras of low cohomological dimension with respect to Hochschild cohomology.

Exercises 5.11. Exercises on dgas.

(1) Let (A, d) be a dga. Show that the Fedosov product on A is associative.
(2) Show that if A is a finite-dimensional algebra, then RA/IA^n is also finite-dimensional for any $n \geq 1$.
(3) Show that (6) is a well-defined coproduct in the category of algebras.
(4) If A is an object in a category $\mathcal{C}$ and the coproduct object $A \sqcup A$ exists, show that $A \sqcup A$ has an automorphism of order 2.

6. Separable algebras

In this section, we describe algebras of cohomological dimension 0 with respect to Hochschild cohomology.

Definition 6.1. *An algebra A has cohomological dimension n if*

$$H^{n+1}(A, M) = 0$$

for all A–bimodules M, but there exists an A–bimodule M with $H^n(A, M) \neq 0$.

From the definition of H^n, we see that A has cohomological dimension 0 if and only if A is a projective object of the category $A - \text{Bimod}$. We will now interpret this condition in terms of differential forms. Recall from Proposition 4.14 that there is a short exact sequence of A–bimodules

$$0 \to \Omega^1(A) \to A \otimes_k A \to A \to 0. \tag{8}$$

If A is a projective object of $A-\text{Bimod}$, this sequence splits. But also, if the sequence (8) splits, then A is a summand of $A \otimes_k A$, which is a free $A \otimes_k A^{op}$–module. Therefore, A is a projective A–bimodule if and only if (8) splits. This is the case if and only if there exists a bimodule map $s : A \to A \otimes_k A$ with $ms = \mathrm{id}_A$. Such a bimodule map s is determined by the element $s(1) \in A \otimes A$, and this must satisfy $ms(1) = 1$ and $as(1) = s(1)a$ for all $a \in A$. Conversely, if $Z \in A \otimes A$ satisfies $m(Z) = 1$ and $aZ = Za$ for all $a \in A$ then we may define a splitting $s : A \to A \otimes A$ of (8) by $s(a) = aZ$ for $a \in A$.

Definition 6.2. *Given an algebra A, a* separability element *is an element Z of the A–bimodule $A \otimes_k A$ such that $aZ = Za$ for all $a \in A$, and $m(Z) = 1$.*

The above discussion shows that (8) splits if and only if A has a separability element.

Now note that a splitting of (8) is equivalent to giving a bimodule map $p : A \otimes_k A \to \Omega^1(A)$ which satisfies $pj = \mathrm{id}_{\Omega^1(A)}$ where $j : \Omega^1(A) \to A \otimes_k A$ is the map in (8), which is defined by $j(a \otimes b) = a \otimes b - ab \otimes 1$. Such a bimodule map p is determined by $Y = p(1 \otimes 1)$ since $p(a \otimes b) = ap(1 \otimes 1)b$ for all $a, b \in A$. If $a \in A$ then $da = pj(da) = pj(1 \otimes a) = p(1 \otimes a - a \otimes 1) = Ya - aY$, so p gives rise to an element $Y \in \Omega^1(A)$ with $da = [Y, a]$ for all $a \in A$. On the other hand, given such a Y, we may define $p(a \otimes b) = aYb$ and this gives a splitting of (8). Thus, we have proved the following theroem.

Theorem 6.3. *For an algebra A, the following are equivalent.*

1. *A has Hochschild cohomological dimension 0.*
2. *The sequence (8) splits.*
3. *A has a separability element.*
4. *The universal derivation $d : A \to \Omega^1(A)$ is inner.*

Definition 6.4. *An algebra A which has a separability element is called* separable.

We will now show that if $k = \mathbb{C}$, then being separable is equivalent to being semisimple.

Definition 6.5. *An algebra A over an algebraically closed field k is called* semisimple *if A is finite-dimensional and every left A–module is projective.*

The Artin-Wedderburn Theorem states that an algebra A is semisimple if and only if there exist $n_1, n_2, \ldots, n_r$ with

$$A \cong M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k).$$

Proposition 6.6. [CQ95a, Section 4] *If A is a separable algebra over an algebraically closed field k, then A is semisimple.*

Proof. Let $Z = \sum_{i=1}^{n} x_i \otimes y_i \in A \otimes A$ be a separability element. We may choose Z with n as small as possible. Let $V = \mathrm{span}\{x_i\}$. We show that V is a left ideal of A and that A acts faithfully on V. Given

$a \in A$, we have

$$\sum_{i=1}^{n} ax_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i a.$$

The $\{y_i\}$ must be linearly independent. If they were not, then by gathering terms we could write Z as the sum of $\leq n-1$ simple tensors, which contradicts the choice of n. Therefore, there exist $\delta_i \in A^*$, the linear dual of A, with $\delta_i(y_j) = \delta_{ij}$. Therefore, applying $1 \otimes \delta_j$, we have $ax_j = \sum_{r=1}^{n} x_r \delta_j(y_r a)$ for all j. Therefore, $ax_j \in V$. So V is a representation of A.

Now we show that V is faithful. If $ax_i = 0$ for all i, then $\sum_i ax_i \otimes y_i = 0$. Applying m, we have $a = a\sum_i x_i y_i = 0$. Therefore, $A \subset \mathrm{End}(V)$ is finite-dimensional.

Now let M be a left A–module. Since the sequence(8) splits, we have $A \otimes_k A \cong A \oplus \Omega^1(A)$ as A–bimodules. Tensoring on the right with M gives $A \otimes_k M \cong M \oplus (\Omega^1(A) \otimes_A M)$ as left A–modules. But $A \otimes_k M$ is a free module, and therefore M is projective, as required. $\square$

Conversely, a semisimple algebra A is always separable. To see this, note that $A = M_{n_1}(k) \times \cdots \times M_{n_r}(k)$ is Morita equivalent to the commutative algebra $S = k \times k \cdots \times k$. But every bimodule over S is projective, because $S \otimes_k S^{op} = S \otimes_k S = \bigoplus_{i,j} U_{ij}$, where the $U_{ij} = ke_i \otimes e_j$ are one-dimensional S–bimodules. Any S–bimodule may be written as a sum of the U_{ij}, and hence is projective. By Morita equivalence, every A–bimodule is also projective, and so A is separable.

Thus, over $\mathbb{C}$ the notions of separable and semisimple are the same.

7. Quasi-free algebras

Now we look at cohomological dimension 1.

In order to understand the condition $H^2(A, M) = 0$ for all M, we begin with a description of $H^2(A, M)$ in terms of extensions. We define $\mathcal{E}xt^2(A, M)$ to be the set of all extensions

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

where:

- B is a k–algebra.
- $\pi : B \to A$ is a surjective algebra map.
- $i(M) \subset B$ is a square-zero ideal.

modulo the equivalence relation that two such extensions $0 \to M \to B \to A \to 0$ and $0 \to M \to B' \to A \to 0$ are equivalent if and only if there is an isomorphism of algebras $\phi : B \to B'$ such that the following diagram commutes.

$$\begin{tikzcd}[row sep=small]
 & & B \arrow[dd, "\phi"] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\
 & & B' \arrow[ur] & &
\end{tikzcd}$$

Note that if $\pi : B \twoheadrightarrow A$ is a surjective algebra map with $\ker(\pi)^2 = 0$, then $\ker(\pi)$ is automatically an A–bimodule. Indeed, a left and right action of A on $\ker(\pi)$ may be defined by taking any vector space splitting $s : A \to B$ of π and setting

$$a \cdot m \cdot b = s(a)ms(b)$$

for $a, b \in A$ and $m \in \ker(\pi)$. It is an exercise to check that this is a well-defined bimodule structure on $\ker(\pi)$. In the definition of the set $\mathcal{E}xt^2(A, M)$, we should therefore make the further requirement that

- The natural A–bimodule structure on $\ker(\pi)$ coincides via i with the bimodule structure on M.

Theorem 7.1. *Let A be a k–algebra and M an A–bimodule. There is a bijection*

$$H^2(A, M) \leftrightarrow \mathcal{E}xt^2(A, M).$$

Proof. See [Lod92, Theorem 1.5.4] for the details. □

We are interested in the situation when $H^2(A, M) = 0$, so that $\mathcal{E}xt^2(A, M)$ has only one element. But there is always a trivial extension, given by $A \oplus M$ with the product $(a_1, m_1)(a_2, m_2) = (a_1a_2, m_1a_2 + a_1m_2)$. This is also denoted $A \ltimes M$. To state that $H^2(A, M) = 0$ is therefore to state that every square-zero extension with kernel M is equivalent to the trivial one. We call such an extension *trivial.*

Lemma 7.2. *An extension*

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

of A by M is trivial if and only if there exists an algebra homomorphism $\ell : A \to B$ such that $\pi\ell = \mathrm{id}_A$.

Proof. If

$$0 \to M \to B \to A \to 0$$

is trivial then there is a $\phi : B \to A \oplus M$ making the following diagram commute.

$$\begin{tikzcd}[row sep=small]
 & & B \arrow[dd, "\phi"] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\
 & & M \oplus A \arrow[ur] & &
\end{tikzcd}$$

If we let $i_A : A \to M \oplus A$ be the insertion, then $\ell := \phi^{-1} i_A$ splits the map $B \to A$, and it is an algebra map because i_A is.

Conversely, suppose such an algebra map $\ell : A \to B$ exists. We may then define $\phi : M \oplus A \to B$ by $\phi(m, a) = j(m) + \ell(a)$ where $j : M \to B$ is the given inclusion map. It is an exercise to check that ϕ is an algebra isomorphism. $\square$

Definition 7.3. *If $\pi : B \twoheadrightarrow A$ is a square-zero extension (ie. an algebra homomorphism with* $\ker(\pi)^2 = 0$*), then an algebra homomorphism $\ell : A \to B$ with $\pi\ell = \mathrm{id}_A$ is called a* lifting homomorphism*.*

We see that $H^2(A, M) = 0$ for all M if and only if every square-zero extension of A has a lifting homomorphism. Now we give another characterization.

Proposition 7.4. [CQ95a, Proposition 3.3] *Let A be an algebra. Then $H^2(A, M) = 0$ for every A–bimodule M if and only if $\Omega^1(A)$ is a projective object in* $A - \mathrm{Bimod}$*.*

Proof. Once again, we use the short exact sequence (8)

$$0 \to \Omega^1(A) \to A \otimes A \to A \to 0.$$

For each i, this yields a long exact sequence of Ext groups in $A - \mathrm{Bimod}$.

$$\cdots \to \mathrm{Ext}^i(A \otimes A, M) \to \mathrm{Ext}^i(\Omega^1(A), M) \to \mathrm{Ext}^{i+1}(A, M) \to \mathrm{Ext}^{i+1}(A \otimes A, M) \to \cdots$$

Since $A \otimes A$ is a free A–bimodule, we get

$$\mathrm{Ext}^i(\Omega^1(A), M) \cong \mathrm{Ext}^{i+1}(A, M)$$

for each i. But $H^i(A, M) = \mathrm{Ext}^i(A, M)$ by definition. Taking $i = 1$, this proves the proposition. $\square$

Definition 7.5. *An algebra A is called* formally smooth *or* quasi-free *if $H^2(A, M) = 0$ for all A–bimodules M.*

We have already seen that formal smoothness is equivalent to every square-zero extension having a lifting homomorphism. More generally, we have the following.

Definition 7.6. *An algebra A is formally smooth if and only if for every square-zero extension $B \to C$ of an algebra C, and every morphism $f : A \to C$, there exists a lifting $\widehat{f} : A \to B$ making the following diagram commute.*

$$\begin{tikzcd} & C \arrow[d] \\ A \arrow[ur] \arrow[r] & B \end{tikzcd}$$

Proof. The proof was shown to me by Yuri Berest. Given a square zero extension $\pi : B \to C$ and a map $f : A \to C$, we form the *pullback* $Z = \{(a, b) \in A \times B : \pi(b) = f(a)\}$.

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The kernel of p is $\{(0, b) : \pi(b) = 0\}$, which is a square-zero ideal in Z because $\ker(\pi)$ is a square-zero ideal in B. Therefore, p has a lifting homomorphism ℓ. Then $\widehat{f} = q\ell$ is the desired lifting of f. □

Proposition 7.7. *An algebra A is quasi-free if and only if for every surjective algebra homomorphism $\pi : B \to C$ with $\ker(\pi)$ a nilpotent ideal, and every $f : A \to C$, there exists a lifting $\widehat{f} : A \to B$.*

Proof. Let $I = \ker(\pi)$. The proof precedes by induction on the nilpotency degree of I. We have a map $f : A \to C/I$, and there is a square-zero extension

$$0 \to I/I^2 \to B/I^2 \to B/I \to 0,$$

so we get a lifting of f to a map $A \to B/I^2$. Now consider the square-zero extension

$$0 \to I^2/I^3 \to B/I^3 \to B/I^2 \to 0$$

and continue in the same manner. □

Proposition 7.7 shows that the Cuntz-Quillen definition of formal smoothness coincides with Grothendieck's definition of formal smoothness in the category of *commutative* rings (see [Gro67, 17.1.1]). It is natural to ask whether a smooth commutative algebra is actually formally smooth in the noncommutative sense. But in fact, this is not the case.

Example 7.8. Let $A = k[x, y]$. Then A is a smooth commutative algebra, but the HKR Theorem states that $H^2(A, A) = \bigwedge_A^2 \operatorname{Der}(A)$, which is a free A–module of rank 1. So A is not quasi-free.

However, $k[x]$ is formally smooth. Indeed, any free algebra is formally smooth because it automatically satisfies any of the stated lifting properties (this is the reason for the terminology quasi-free). We shall see later that formally smooth algebras are rather scarce.

The fact that $k[x, y]$ is not formally smooth means that there is some square-zero extension of $k[x, y]$ for which no lifting homomorphism exists. If we want to construct an explicit example of such an extension, a first guess would be to take

$$k\langle X, Y\rangle/\langle XY - YX\rangle^2 \to k[x, y].$$

To show that there is no lifting homomorphism for this extension, we can use the following proposition.

Proposition 7.9. *Let A be a k–algebra. The following are equivalent.*

(1) *A is formally smooth.*

(2) *For all presentations $A = F/I$ with F free, there exists a lifting homomorphism $F/I \to F/I^2$.*

(3) *There exists a presentation $A = F/I$ with F free, such that there is a lifting homomorphism $F/I \to F/I^2$.*

Proof. We just need to check that the existence of a presentation $A = F/I$ with a lifting homomorphism $\zeta : F/I \to F/I^2$ implies that A is quasi-free. For this, let $\pi : B \twoheadrightarrow A$ be any square-zero extension. We need to find a lifting homomorphism $\ell : A \to B$.

Suppose F is the free algebra on $\{X_i : i \in I\}$ and write x_i for $X_i + I \in A$. For each $i \in I$, choose $b_i \in B$ with $\pi(b_i) = x_i$. Define $\lambda : F \to B$ by $\lambda(X_i) = b_i$. Then $\pi\lambda$ is the quotient map $F \to F/I$, and hence $\ker(\pi\lambda) = I$. Therefore, $\lambda(I) \subset \ker(\pi)$ and so $\lambda(I^2) \subset \lambda(I)^2 = 0$. Therefore λ induces a map $\overline{\lambda} : F/I^2 \to B$. Taking $\ell = \overline{\lambda}\zeta$, we get $\pi\ell = \pi\overline{\lambda}\zeta = \mathrm{id}_A$, as required. □

7.1. **The universal extension.** Recall from Proposition 4.13 that for an algebra A, $RA = R_kA$ is the free algebra on the vector space $\overline{A}$. Thus, Proposition 7.9 implies in particular that A is quasi-free if and only if the square-zero extension

$$RA/IA^2 \to A$$

has a lifting homomorphism. By analysing such lifting homomorphisms more closely, Cuntz and Quillen related quasi-freeness to existence of a *right connection* $\nabla : \Omega^1(A) \to \Omega^2(A)$. We now explain this.

Proposition 5.7 implies that RA/IA^2 is the vector space $A \oplus \Omega^2(A)$ equipped with the Fedosov product. Any lifting homomorphism $A \to RA/IA^2$ is given by $a \mapsto a - \phi(a)$ for some linear map $\phi : \overline{A} \to \Omega^2(A)$. The condition that $a \mapsto a - \phi(a)$ is an algebra map implies

$$(a_1 - \phi(a_1)) \circ (a_2 - \phi(a_2)) = a_1 \circ a_2 - \phi(a_1) \circ a_2 - a_1 \circ \phi(a_2) + \phi(a_1) \circ \phi(a_2)$$

for all $a_1, a_2 \in A$, where $\circ$ is the Fedosov product. This reduces to the equation

$$\phi(a_1a_2) = a_1\phi(a_2) + \phi(a_1)a_2 + da_1da_2 \tag{9}$$

for all $a_1, a_2 \in A$.

We relate (9) to a splitting of a short exact sequence. Recall that there is a short exact sequence

$$0 \to \Omega^1(A) \to A \otimes A \to A \to 0$$

of A–bimodules. As a sequence of left A–modules, this sequence splits because A is a free left A–module. We can then tensor on the left with $\Omega^1(A)$ to obtain another split short exact sequence of the form

$$0 \longrightarrow \Omega^1(A) \otimes_A \Omega^1(A) \longrightarrow \Omega^1(A) \otimes_k A \longrightarrow \Omega^1(A) \longrightarrow 0$$

But it follows from the proof of Proposition 5.3 that $\Omega^1(A) \otimes_A \Omega^1(A) \cong \Omega^2(A)$, and so we have a short exact sequence of A–bimodules

$$0 \longrightarrow \Omega^2(A) \xrightarrow{j} \Omega^1(A) \otimes_k A \xrightarrow{m} \Omega^1(A) \longrightarrow 0 \tag{10}$$

where $j(a_0 da_1 da_2) = a_0 da_1 da_2 \otimes 1 - a_0 da_1 \otimes da_2$ and $m(a_0 da_1 \otimes b) = a_0 da_1 \cdot b$.

Lemma 7.10. [CQ95a, Proposition 3.4] *There exists a linear map $\phi : \overline{A} \to \Omega^2(A)$ satisfying (9) if and only if the short exact sequence (10) splits.*

Proof. A splitting of (10) is equivalent to a map of A–bimodules $p : \Omega^1(A) \otimes A \to \Omega^2(A)$ with $pj = \mathrm{id}$. Since $\Omega^1(A) \otimes_k A = A \otimes_k \overline{A} \otimes_k A$, such a p is of the form $p(axb) = a\phi(x)b$ for some $\phi : \overline{A} \to \Omega^2(A)$. By evaluating p at $da_1 da_2$ for $a_1, a_2 \in A$, we get

$$\begin{aligned} p(da_1 da_2) &= p(da_1 \cdot a_2 \otimes 1 - da_1 \otimes a_2 \\ &= p(d(a_1 a_2) \otimes 1 - a_1 da_2 \otimes 1 - da_1 \otimes da_2) \\ &= \phi(a_1 a_2) - a_1 \phi(a_2) - \phi(a_1) a_2. \end{aligned}$$

From this, we see that $pj = \mathrm{id}$ is equivalent to ϕ satisfying (9). $\square$

We can use this to get another characterization of quasi-freeness.

Proposition 7.11. [CQ95a, Proposition 3.4] *A k–algebra A is quasi-free if and only if there exists a linear map*

$$\nabla_r : \Omega^1(A) \to \Omega^2(A)$$

satisfying

$$\nabla_r(a\omega) = a\nabla_r(\omega)$$

$$\nabla_r(\omega a) = (\nabla_r \omega)a + \omega da$$

for all $\omega \in \Omega^1(A)$ and all $a \in A$.

Proof. From Lemma 7.10, we see that A is quasi-free if and only if there exists $\phi : \overline{A} \to \Omega^2(A)$ satisfying (9). Giving a linear map $\phi : \overline{A} \to \Omega^2(A)$ is the same as giving a left A–map $\nabla_r : A \otimes_k \overline{A} = \Omega^1(A) \to \Omega^2(A)$, defined by $\nabla_r(a \otimes b) = a\phi(b)$. We then compute

$$\nabla_r(a_0 da_1 a) - \nabla_r(a_0 da_1)a - a_0 da_1 da = \nabla_r(a_0(da_1 a)) - \nabla_r(a_0 a_1 da) - \nabla_r(a_0 da_1)a - a_0 da_1 da$$

which equals

$$a_0 \phi(a_1 a) - a_0 a_1 \phi(a) - a_0 \phi(a_1) a - a_0 da_1 da$$

for all $a_0, a_1, a \in A$. Therefore, (9) is equivalent to

$$\nabla_r(a_0 da_1 a) = \nabla_r(a_0 da_1)a + a_0 da_1 a$$

for all $a_0, a_1, a \in A$. $\square$

Remark 7.12. A map of the form ∇_r is called a *right connection* on the bimodule $\Omega^1(A)$. Thus, we can say that A is quasi-free if and only if $\Omega^1(A)$ has a right connection.

7.2. **Examples of quasi-free algebras.** So far, the only examples we have seen of quasi-free algebras are free algebras and semisimple algebras. We now give some further examples.

The first thing we want to prove is that Hochschild dimension ≤ 1 implies global dimension ≤ 1. Recall that an algebra is called (left) *hereditary* if every submodule of a projective (left) module is projective. Equivalently, every module has a projective resolution of length 1, that is, if M is a left A–module then there is a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1 projective.

Theorem 7.13. *If A is a quasi-free algebra then A is left and right hereditary.*

Proof. We begin by considering the short exact sequence (8) of A–bimodules.

$$0 \to \Omega^1(A) \to A \otimes_k A \to A \to 0$$

This splits as a sequence of *right* A–modules, because A is a projective right A–module. Therefore, $A \otimes_k A \cong \Omega^1(A) \oplus A$ as right A–modules. Now let M be a left A–module. Then we have $A \otimes_k M \cong (\Omega_1(A) \otimes_A M) \oplus M$, and hence the sequence obtained by tensoring (8) with M on the right remains exact. Thus, we have a short exact sequence of left A–modules

$$0 \to \Omega^1(A) \otimes_A M \to A \otimes_k M \to M \to 0.$$

The middle term is a projective left A–module because it is a direct sum of copies of A. We show that the left hand term is projective. Because A is quasi-free, $\Omega^1(A)$ is a projective A–bimodule. Therefore, there exists an A–bimodule Q with

$$\Omega^1(A) \oplus Q \cong (A \otimes_k A^{op}) \otimes_k B$$

for some vector space B (B is just a convenient way of recording how many copies of $A \otimes A^{op}$ appear in the free module). Applying $- \otimes_A M$ we obtain

$$(\Omega^1(A) \otimes_A M) \oplus (Q \otimes_A M) \cong A \otimes_k B \otimes_k M$$

and so $\Omega^1(A) \otimes_A M$ is a summand of a free left A–module, so it is projective. Thus, M has a projective resolution of length ≤ 1. The same argument works when M is a right A–module, showing that A is left and right hereditary. $\square$

Theorem 7.13 imposes a strong restriction on a quasi-free algebra A. For example, if A is a finite-dimensional k–algebra with k algebraically closed, then A is hereditary if and only if A is Morita equivalent to a path algebra kQ for some quiver Q. If A is a finitely-generated commutative k–algebra, then A hereditary implies that A has finite global dimension, so A must be smooth, and furthermore A must have Krull dimension 1. Thus, A is the coordinate ring of a smooth affine curve.

Having seen that the class of quasi-free algebras is quite restricted, let us now list some ways to construct new quasi-free algebras from old ones.

Proposition 7.14. [CQ95a, 5.3] *If A is a quasi-free algebra, then the following algebras are also quasi-free.*

(1) *The free product $A * B$, for any quasi-free algebra B.*
(2) *Any formal localization A_S for $S \subset A$.*
(3) *$T_A(N)$ where N is any projective object in $A - \mathrm{Bimod}$.*
(4) *Any algebra C which is Morita equivalent to A.*

Proof. (1) If A and B are quasi-free k–algebras, suppose $\pi : C \to A * B$ is a surjective algebra map with $\ker(\pi)^2 = 0$. We wish to find a lifting homomorphism $\ell : A * B \to C$. There are natural maps $A \to A * B$ and $B \to A * B$ into the coproduct. We get lifting homomorphisms $\ell_A : A \to C$ and $\ell_B : B \to C$. These combine to give the desired lifitng homomorphism $(\ell_A, \ell_B) : A * B \to C$.

(2) For a subset $S \subset A$, the algebra A_S is defined via the following universal property: there is a map $\alpha : A \to A_S$ such that $\alpha(s)$ is a unit for all $s \in S$, and if $\theta : A \to B$ is any algebra map such that $\theta(s)$ is a unit for all $s \in S$, then there exists a unique $\theta' : A_S \to B$ such that $\theta'\alpha = \theta$, in other words, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr] & A_S \arrow[d] \\ & B \end{tikzcd}$$

The existence of A_S may be proved by taking $A_S = A * k\langle t_s : s \in S\rangle / I$, where I is the two sided ideal generated by $t_s s - 1$ and $s t_s - 1$ for $s \in S$.

Now suppose A is quasi-free. We wish to show that A_S is quasi-free. Suppose $\pi : B \to A_S$ is a surjective algebra map with $\ker(\pi)^2 = 0$. Then the natural map $\alpha : A \to A_S$ has a lifting $\theta : A \to B$ with $\pi\theta = \alpha$. If we can show that $\theta(s)$ is a unit for all $s \in S$, then the universal property of A_S will give the desired lifting $A_S \to B$. Let $s \in S$. Since $\pi\theta(s)$ is a unit, there exists $u \in B$ with $1 - u\theta(s), 1 - \theta(s)u \in \ker(\pi)$. Therefore, $(1 - u\theta(s))^2 = (1 - \theta(s)u)^2 = 0$. But for any u, v, we have $(1 - uv)^2 = 1 - 2uv + uvuv = 1 - (2u - uvu)v$ and so $\theta(s)$ has a left inverse in B. Similarly, $\theta(s)$ has a right inverse, so the left and right inverses must coincide, and $\theta(s)$ is a unit.

(3) Suppose A is a quasi-free algebra and N is a projective A–bimodule. Suppose $\pi : B \to T_A(N)$ is a surjective algebra map with $\ker(\pi)^2 = 0$. There is a natural map $A \to T_A(N)$ and this induces a

map $\theta : A \to B$ because A is quasi-free. The map θ makes B into an A–bimodule. Thus, because N is projective and we have a surjection $\pi : B \to T_A(N)$ of A–bimodules, we get a map of A–bimodules $\psi : N \to B$, by definition of projectivity. We have the following commutative diagram of A–bimodule maps.

$$\begin{tikzcd}
 & N \arrow[dl, "\psi"'] \arrow[d] \\
B \arrow[r, "\pi"] & T_A(N) \\
 & A \arrow[ul, "\theta"] \arrow[u]
\end{tikzcd}$$

The universal property of $T_A(N)$ then gives a map $T_A(N) \to B$ which is a lifting homomorphism for π (recall that this map is given explicitly by $n_1 \otimes \cdots \otimes n_r \mapsto \psi(n_1) \cdots \psi(n_r)$ in degree ≥ 1).

(4) If A and B are Morita equivalent, then Theorem 3.9 implies that $H^2(A, M) = 0$ for all M if and only if $H^2(B, M) = 0$ for all M. Thus, A is quasi-free if and only if B is quasi-free.

□

Examples 7.15. Some examples:

(1) If A is a quasi-free algebra, then by Proposition 7.14, so is $M_n(A)$, because it is Morita equivalent to A.
(2) If Q is a quiver then $kQ = T_S(A)$ by definiton, where A is the span of the arrows of Q. The S–bimodule A is projective because S is semisimple, and so kQ is quasi-free.
(3) If A is a quasi-free algebra and a finite group G acts on A by algebra automorphisms, then $T_{kG}(A)$ is quasi-free.

Exercises 7.16. Exercises on quasi-freeness.

(1) Let R be a ring. Show that every left R–module has a projective resolution of length ≤ 1 if and only if every submodule of a projective module is projective. (Hint: use the long exact sequence for Ext).
(2) Show that if A is quasi-free, then so is the algebra ΩA.
(3) If A and B are quasi-free algebras, show that the product $A \times B$ is quasi-free.
(4) Show that the tensor product $A \otimes_k B$ of quasi-free algebras need not be quasi-free.

7.3. **Quasi-free algebras and completions.** Let R be a k–algebra and $I \subset R$ a two-sided ideal. Then we may define the *i–adic completion of R with respect to I* as follows.

$$\widehat{R} = \varprojlim_{n \geq 1} R/I^n$$

Explicitly, this is the set of sequences

$$\{(s_1, s_2, \ldots) : s_i \in R/I^i \text{ and } s_{i+1} + I^i = s_i \text{ for all } i\}.$$

Recall that this algebra may also be defined via the universal property that for all i, there exists an algebra homomorphism $\pi_i : \widehat{R} \to R/I^i$, such that the following diagram commutes.

$$\begin{tikzcd} & \widehat{R} \arrow[dl] \arrow[dr] & \\ R/I^{i+1} \arrow[rr] & & R/I^i \end{tikzcd}$$

for all i, and such that if Z is a k–algebra and there are maps $\theta_i : Z \to R/I^i$ for all i making the following diagram commute

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ R/I^{i+1} \arrow[rr] & & R/I^i \end{tikzcd}$$

then there exists a unique map $\theta : Z \to \widehat{R}$ with $\pi_i\theta = \theta_i$ for all i.

We can characterize quasi-freeness in terms of completions in the following way.

Proposition 7.17. *Let A be a k–algebra. Then A is quasi-free if and only if for all k–algebras R and ideals $I \subset R$, if $\alpha : A \to R/I$ is an algebra map, then there exists an extension $\widehat{\alpha} : A \to \widehat{R}$ such that the diagram*

$$\begin{tikzcd} & \widehat{R} \arrow[d, "\pi_1"] \\ A \arrow[ur, "\widehat{\alpha}"] \arrow[r, "\alpha"'] & R/I \end{tikzcd}$$

commutes.

Proof. Suppose the given lifting property holds. Let $\pi : B \to A$ be a square-zero extension and let $I = \ker(\pi)$. Denote by $\widehat{B}$ the completion of B with respect to the ideal I. Then $\widehat{B} \cong B$ and therefore there exists a lifting homomorphism $A \to B$, so A is quasi-free.

Conversely, suppose A is quasi-free. Let $\alpha : A \to R/I$. By quasi-freeness, this can be lifted to $\alpha_2 : A \to R/I^2$. Inductively, as in the proof of Proposition 7.7, we obtain $\alpha_i : A \to R/I^i$ for every $i \geq 1$, and therefore a lifting homomorphism $\widehat{\alpha} : A \to \widehat{R}$. $\square$

One of the things proved by Cuntz and Quillen in [CQ95a, Section 7] is that given an algebra homomorphism $\alpha : A \to R/I$, there is a canonical way to get a lifting $A \to \widehat{R}$. We will explain this construction and also explain in what sense it is universal.

Suppose A is a quasi-free k–algebra. Write RA for R_kA. There exists a lifting homomorphism $A \to RA/IA^2$. We will use this to construct a map $\ell : A \to \widehat{RA}$, where $\widehat{RA}$ denotes the completion of RA with respect to the ideal IA. Recall from Section 7.1 that the lifting $A \to RA/IA^2$ has the form $a \mapsto a - \phi a$ where $\phi : \overline{A} \to \Omega^2(A)$ satisfies

$$a_1\phi(a_2) + \phi(a_1)a_2 = \phi(a_1a_2) - da_1da_2$$

for all $a_1, a_2 \in A$. By Lemma 4.13, RA is the free algebra $T_k(\overline{A})$. Thus, we may define a *derivation* $D : RA \to RA$ by setting

$$Da = \phi(a)$$

for $a \in \overline{A}$. This makes sense because $\phi(a) \in \Omega^2(A)$, and we have an identification $RA \cong (\Omega^{ev}(A), \circ)$ where $\circ$ denotes the Fedosov product of forms. From Proposition 5.7, we also know that under this identification, IA^k is identified with the even forms of degree $\geq 2k$. Since $D(1) = 0$ because D is a derivation, we have $D(RA) \subset \Omega^2(A) \subset IA$ and therefore $D(IA^k) \subset IA^k$ for every $k \geq 1$.

Following [CQ95a], we now make the following calculation. For $a_1, a_2 \in A$, we have

$$\begin{aligned} D(da_1 da_2) &= D(a_1 a_2 - a_1 \circ a_2) \\ &= \phi(a_1 a_2) - a_1 \circ D(a_2) - D(a_1) \circ a_2 \\ &= \phi(a_1 a_2) - a_1 \phi(a_2) - \phi(a_1) a_2 + da_1 d\phi(a_2) + d\phi(a_1) da_2 \\ &= da_1 da_2 + da_1 d\phi(a_2) + d\phi(a_1) da_2 \end{aligned}$$

Now consider the action of D on a $2n$–form $a_0 da_1 da_2 \cdots da_{2n-1} da_{2n}$. Using the fact that D is a derivation and the above calculation of $D(da_1 da_2)$, we see that

$$D(a_0 da_1 da_2 \cdots da_{2n-1} da_{2n}) = (H + L)(a_0 da_1 da_2 \cdots da_{2n-1} da_{2n})$$

where $H : RA \to RA$ is the linear map defined by $H(\omega) = i\omega$ for $\omega \in \Omega^{2i}(A)$, and where $L : RA \to RA$ is the linear map defined by

$$L(a_0 da_1 da_2 \cdots da_{2n-1} da_{2n}) = \phi(a_0) da_1 da_2 \cdots da_{2n-1} da_{2n} + \sum_{j=1}^{2n} a_0 da_1 \cdots da_{j-1} d\phi(a_j) da_{j+1} \cdots da_{2n}.$$

Thus, by linearity, we have

$$D = H + L.$$

Also, since L raises degree by 2, we obtain that

$$(D - k) \cdots (D - 1) D(RA) \subset IA^{k+1}.$$

Now, for a fixed k, D induces a derivation $\overline{D} : RA/IA^{k+1} \to RA/IA^{k+1}$, and this derivation satisfies $(\overline{D} - k) \cdots (\overline{D} - 1)\overline{D} = 0$. We now use the following exercise.

Exercise 7.18. Let V be a (possibly infinite-dimensional) vector space and $A : V \to V$ a linear transformation. Suppose that $(A - \lambda_1) \cdots (A - \lambda_n) = 0$ for some scalars λ_i, which are all distinct. Then $V = \bigoplus_{i=1}^n \ker(A - \lambda_i)$.

From Exercise 7.18, we conclude that

$$RA/IA^{k+1} = \bigoplus_{i=0}^{k} \ker(\overline{D} - i),$$

the direct sum of the eigenspaces of $\overline{D}$. From Proposition 5.7, we also know that

$$RA/IA^{k+1} = \bigoplus_{i=0}^{k} \Omega^{2i}(A).$$

For an eigenvector v of $\overline{D}$, we have $v = \sum_{i=0}^{k} \omega_i$ with $\omega_i \in \Omega^{2i}(A)$. Define the *leading term* of v to be $\sigma(v) = \omega_i$, with the smallest i such that $\omega_i \neq 0$. Since $D = H + L$, we see that if v is an eigenvector with eigenvalue i, then $\sigma(v) \in \Omega^{2i}(A)$. We claim that the map $\sigma : \ker(\overline{D} - i) \to \Omega^{2i}(A)$ is a vector space isomorphism.

To see that σ is one-to-one, suppose that v and w are i–eigenvectors with the same leading term. Then $D(v - w) = i(v - w) \in \Omega^{2i}(A) \cap \bigoplus_{j>i} \Omega^{2j}(A) = 0$. So $i(v - w) = 0$ and hence $v = w$. Therefore, the map is one-to-one. To see that it is onto, let $\omega \in \Omega^{2i}(A)$. We exhibit an eigenvector v with $\sigma(v) = \omega$ by $v = e^{-L}(\omega)$. Since we are working in RA/IA^{k+1}, L is nilpotent and so e^{-L} makes sense. We may compute:

$$\begin{aligned}
\overline{D}v &= (H + L)e^{-L}(\omega) \\
&= H \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L^k \omega + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L^{k+1} \omega \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (k + i) L^k \omega + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L^{k+1} \omega \\
&= i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L^k \omega \\
&= iv
\end{aligned}$$

Thus, $e^{-L} : \Omega^{2i}(A) \to \ker(\overline{D} - i)$ is a linear isomorphism which is inverse to σ, since clearly $\sigma(e^{-L}\omega) = \omega$. Now, e^{-L} is also an algebra isomorphism, because if $\omega \in \Omega^{2i}(A)$ and $\eta \in \Omega^{2j}(A)$ are homogeneous elements, then $e^{-L}(\omega\eta)$ and $e^{-L}(\omega)e^{-L}(\eta)$ are eigenvectors of $\overline{D}$ with eigenvalue $i+j$ and leading term $\omega\eta$. Therefore, they must be equal. Hence, e^{-L} is a bijective algebra map and therefore is an algebra isomorphism.

Corollary 7.19. *If A is quasifree then for any $k \geq 0$, there is an isomorphism of algebras*

$$e^{-L} : \bigoplus_{i=0}^{k} \Omega^{2i}(A) \to RA/IA^{k+1}.$$

Combining Corollary 7.19 with Proposition 5.7 shows that the vector space $\bigoplus_{i=0}^{k} \Omega^{2i}(A)$ equipped with the Fedosov product is isomorphic to the same vector space equipped with the usual product of forms.

Combining the isomorphims e^{-L} for different values of k yields the following corollary.

Corollary 7.20. [CQ95a, Section 7] *If A is quasifree then there is an isomorphism of algebras*

$$e^{-L} : \widehat{\Omega}^{ev}(A) \to \widehat{RA}$$

where the left hand side is the completion of $\Omega^{ev}(A)$ with respect to the ideal of forms of positive degree, and the right hand side is the completion of RA with respect to the ideal IA.

7.4. **Universal lifting homomorphism.** We are now in a position to answer the question from the beginning of this section. Let A be a quasifree algebra and let R be an arbitrary algebra with $I \subset R$ a two-sided ideal. Suppose we are given a map $\alpha : A \to R/I$. We show how to extend α to $\widehat{\alpha} : A \to \widehat{R}$ in such a way that the following diagram commutes.

$$\begin{tikzcd} & \widehat{R} \arrow[d, "\pi_1"] \\ A \arrow[ur, "\widehat{\alpha}"] \arrow[r, "\alpha"] & R/I \end{tikzcd}$$

Since A is quasifree, there exists $\phi : \overline{A} \to \Omega^2(A)$ such that $a \mapsto a - \phi(a)$ is a lifting homomorphism $A \to RA/IA^2$. Using the definitions from the previous section, we obtain a lifting homomorphism $e^{-L} : A \to \widehat{RA}$. For any $k \geq 1$, this induces a homomorphism $A \to RA/IA^{k+1}$, which we also denote by e^{-L}.

Given our map $\alpha : A \to R/I$, let $n \geq 2$. We first find a map $\alpha_n : A \to R/I^n$ making the following diagram commute.

$$\begin{tikzcd} & R/I^n \arrow[d, "\pi"] \\ A \arrow[ur, "\alpha_n"] \arrow[r, "\alpha"] & R/I \end{tikzcd}$$

We will then put the α_n together to construct the map $\widehat{\alpha}$.

Note that $\ker(\pi) = I/I^n$, so $\ker(\pi)^n = 0$. Let $s_n : R/I \to R/I^n$ be a linear map with $\pi s_n = \mathrm{id}$ and with $s_n(1) = 1$. We may choose the s_n for each $n \geq 2$ in a compatible way, that is, s_{n-1} is the composition of the canonical projection $R/I^n \to R/I^{n-1}$ with $s_n : R/I \to R/I^n$. For $x, y \in A$, we have $s_n\alpha(xy) - s_n\alpha(x)s_n\alpha(y) \in \ker(\pi)$, and so $s_n\alpha : A \to R/I^n$ is a based linear map whose n^{th} curvature vanishes. Therefore, there exists a unique algebra map $\psi_n : RA/IA^n \to R/I^n$ defined by $\psi_n(a) = s_n\alpha(a)$ for $a \in \overline{A}$. But we also have the ring homomorphism $e^{-L} : A \to RA/IA^n$, and we now claim that $\psi_n e^{-L}$ lifts α. That is, we claim that the following square commutes.

$$\begin{tikzcd} RA/IA^n \arrow[r, "\psi_n"] & R/I^n \arrow[d, "\pi"] \\ A \arrow[u, "e^{-L}"] \arrow[r, "\alpha"] & R/I \end{tikzcd}$$

To see this, let $a \in A$. Then $e^{-L}(a) = a + z$ for some $z \in \bigoplus_{j=1}^n \Omega^{2j}(A) = IA/IA^n \subset RA/IA^n$. Therefore, $\pi\psi_n e^{-L}(a) = \pi\psi_n(a+z) = \pi s_n\alpha(a) + \pi\psi_n(z) = \alpha(a) + \pi\psi_n(z)$. We need to show that $\pi\psi_n(IA) = 0$. But $IA \subset RA$ is generated by elements of the form $x \otimes y - xy$ for $x, y \in A$. Since $\pi\psi_n$ is a ring homomorphism, we have

$$
\begin{aligned}
\pi\psi_n(x\otimes y - xy) &= \pi\psi_n(x)\pi\psi_n(y) - \pi\psi_n(xy) \\
&= \pi s_n\alpha(x)\pi s_n\alpha(y) - \pi s_n\alpha(xy) \\
&= \alpha(x)\alpha(y) - \alpha(xy) \\
&= 0
\end{aligned}
$$

for all $x, y \in A$. We obtain $\pi\psi_n e^{-L} = \alpha$ as required. We may therefore take the lift $\alpha_n := \psi_n e^{-L}$.

To solve the original problem, we may put the maps ψ_n together to get a ring homomorphism $\psi : \widehat{RA} \to \widehat{R}$, and we see that the following square commutes.

$$
\begin{tikzcd}
\widehat{RA} \arrow[r, "\psi"] & \widehat{R} \arrow[d, "\pi_1"] \\
A \arrow[u, "e^{-L}"] \arrow[r, "\alpha"] & R/I
\end{tikzcd}
$$

We take $\widehat{\alpha} = \psi e^{-L}$. Thus, using nothing but the original ϕ, we have constructed a lift of an arbitrary morphism $A \to R/I$ to a morphism $A \to \widehat{R}$.

7.5. **A universal property.** The following exercise explains one sense in which the above construction is universal.

Exercise 7.21. Let $A = R/I$ be any nilpotent extension of a quasifree algebra A (ie. I is a nilpotent ideal of R). Let $\rho : A \to R$ be a based linear map such that the composition of $R \to R/I$ with ρ is the identity. Show that there exists a unique algebra map $\rho_* : \widehat{RA} \to R$ such that $\rho_* e^{-L}$ is a lifting homomorphism for the projection $R \to R/I = A$. Here, $\widehat{RA}$ denotes the completion of RA with respect to the ideal IA, as above.

8. Connections

We end these lecture notes with some remarks on connections, which were introduced in Section 7.1 above. The notion of a connection on a one-sided module is due to Connes, and this was generalised to connections on bimodules by Cuntz and Quillen.

8.1. **Connes connections.** Let A be an algebra and let E be a right A–module.

Definition 8.1. *A (Connes)* connection *on E is a linear map*

$$\nabla : E \to E \otimes_A \Omega^1(A)$$

such that

$$\nabla(\xi a) = \nabla(\xi)a + \xi \otimes da$$

for all $a \in A$ and all $\xi \in E$.

Any such ∇ may be extended uniquely to $\nabla : E \otimes_A \Omega A \to E \otimes_A \Omega A$ in such a way that $\nabla(\eta\omega) = (\nabla\eta)\omega + (-1)^{|\eta|}\eta d\omega$ for homogeneous forms $\eta \in E \otimes \Omega A$, $\omega \in \Omega A$ of degrees $|\eta|, |\omega|$ respectively.

There is a natural multiplication map $m : E \otimes_k A \to E$. We claim that sections of the right A–module map m correspond bijectively to connections on E. We now verify this claim.

Recall that we have a short exact sequence of A–bimodules

$$0 \longrightarrow \Omega^1(A) \xrightarrow{j} A \otimes_k A \longrightarrow A \longrightarrow 0.$$

Because A is a projective left A–module, the sequence splits as a sequence of left A–modules. Therefore, on tensoring it on the left by any right A–module E_A, we obtain a short exact sequence of *right* A–modules of the form

$$0 \longrightarrow E \otimes_A \Omega^1(A) \xrightarrow{1\otimes j} E \otimes_k A \xrightarrow{m} E \longrightarrow 0$$

in which the second map is m. Let us temporarily denote by π the natural quotient map $E \otimes_k \Omega^1(A) \to E \otimes_A \Omega^1(A)$.

Given a section s of m, we define $\nabla_s : E \to E \otimes_A \Omega^1(A)$ by $\nabla_s = \pi(1 \otimes d)s$. We now check that ∇_s is a connection on E. Given $\xi \in E$ and $a \in A$, we write $s(\xi) = \sum \xi_1 \otimes \xi_2$ with $\xi_1 \in E$ and $\xi_2 \in A$. We may then make the following calculation.

$$\begin{aligned}
\nabla(\xi a) &= \pi(1 \otimes d)s(\xi)a \\
&= \sum \pi(\xi_1 \otimes d(\xi_2 a)) \\
&= \sum \pi(\xi_1 \otimes (\xi_2 da + d(\xi_2)a)) \\
&= \sum \xi_1\xi_2 \otimes da + \sum (\xi_1 \otimes d\xi_2)a \\
&= ms(\xi) \otimes da + (\nabla_s \xi)a \\
&= \xi \otimes da + (\nabla_s \xi)a
\end{aligned}$$

This shows that ∇_s is a connection on E.

Conversely, given a connection $\nabla : E \to E \otimes_A \Omega^1(A)$, we define

$$s_\nabla(\xi) = \xi \otimes 1_A - (1 \otimes j)(\nabla(\xi))$$

for $\xi \in E$. We now verify that s_∇ is a section of m. Because $m(1 \otimes j) = 0$, all that needs to be checked is that s_∇ is a right A–map. We have

$$\begin{aligned}
s_\nabla(\xi a) &= \xi a \otimes 1_A - (1 \otimes j)(\nabla(\xi)a + \xi \otimes da) \\
&= \xi \otimes a - (1 \otimes j)(\nabla(\xi))a - \xi \otimes a + \xi a \otimes 1
\end{aligned}$$

for all $\xi \in E$ and all $a \in A$, where we used the definition of j from Proposition 4.14. The last two terms cancel, which shows that s is an A–module map, as desired.

We now check that $\nabla \mapsto s_\nabla$ and $s \mapsto \nabla_s$ are mutually inverse. Given a connection ∇ on E, let us write $\nabla\xi = \sum \gamma_1 \otimes d\gamma_2$ for $\xi \in E$. We then calculate as follows.

$$
\begin{aligned}
\nabla_{s_\nabla}(\xi) &= \pi(1 \otimes d)s_\nabla(\xi) \\
&= \pi(1 \otimes d)(\xi \otimes 1_A - j(\nabla\xi)) \\
&= \pi(1 \otimes d)(-\sum j(\gamma_1 \otimes d\gamma_2)) \\
&= \sum \pi(1 \otimes d)(-\gamma_1\gamma_2 \otimes 1 + \gamma_1 \otimes \gamma_2) \\
&= \sum \gamma_1 \otimes d\gamma_2 \\
&= \nabla(\xi).
\end{aligned}
$$

Finally, given a splitting s of m, we write $s(\xi) = \sum \xi_1 \otimes \xi_2$ as before, and calculate as follows.

$$
\begin{aligned}
s_{\nabla_s}(\xi) &= \xi \otimes 1_A - j(\nabla_s(\xi)) \\
&= \xi \otimes 1_A - j\pi(1 \otimes d)\sum \xi_1 \otimes \xi_2 \\
&= \xi \otimes 1_A - \sum(\xi_1\xi_2 \otimes 1_A - \xi_1 \otimes \xi_2) \\
&= \xi \otimes 1_A - ms(\xi) \otimes 1_A + \sum \xi_1 \otimes \xi_2 \\
&= s(\xi),
\end{aligned}
$$

so that $s_{\nabla_s} = s$.

This finishes the proof that $s \mapsto \nabla_s$ is a bijection between the set of splittings of m and the set of connections on E.

Corollary 8.2 (Connes)**.** *A right A–module E_A has a connection if and only if E_A is projective.*

Proof. If E_A has a connection, then $E \otimes_k A \to E \to 0$ splits as a sequence of right A–modules. But then E is a summand of the free right A–module $E \otimes_k A$, so is projective. Conversely, if E is projective then m is split, and so E must have a connection. $\square$

Examples 8.3. It is easy to check that $d : A \to \Omega^1(A)$ is a connection on A_A, for any algebra A. Generalising this, there is a connection on A^n_A given by $\nabla(a_1, \dots, a_n) = (da_1, \dots da_n)$. Finally, for a general finitely-generated projective right A–module P, we have $P = eA^n$ for some idempotent $e \in \mathrm{End}_A(A^n)$. Then we may define a connection in P by sending $e(a_1 \dots, a_n)$ to $e(da_1, \dots, da_n)$. This is called a *Grassmannian connection* in [CQ95a, Section 8].

8.2. **Right and left connections.** In the paper [CQ95a], the above notions of Connes were generalised to bimodules E. In the following, let A be an algebra and let ${}_AE_A$ be a bimodule.

Definition 8.4. *A* right connection *on E is a linear map*

$$\nabla_r : E \to E \otimes_A \Omega^1(A)$$

such that for all $a \in A$ and all $\xi \in E$, we have

$$\nabla_r(a\xi) = a\nabla_r(\xi)$$

$$\nabla_r(\xi a) = (\nabla_r\xi)a + \xi \otimes da.$$

A left connection *on E is a linear map*

$$\nabla_\ell : E \to \Omega^1(A) \otimes_A E$$

such that for all $a \in A$ and all $\xi \in E$, we have

$$\nabla_\ell(\xi a) = \nabla_\ell(\xi)a$$

$$\nabla_\ell(a\xi) = a(\nabla_\ell\xi) + da \otimes \xi.$$

In particular, we see that a right connection on ${}_AE_A$ is a special kind of Connes connection on E_A. We can relate the existence of a right connection to the existence of a splitting in the same way as we did for Connes connections. To do this, let ${}_AE_A$ be an A–bimodule and let $\nabla : E \to E \otimes_A \Omega^1(A)$ be a Connes connection on E_A. Then the definition of s_∇ leads to the following formula.

$$s_\nabla(a\xi) - as_\nabla(\xi) = (1 \otimes j)(-\nabla(a\xi) + a\nabla(\xi))$$

for all $a \in A$ and for all $\xi \in E$. From this we see that if ∇ is a left A–map, then so is s_∇. Conversely, if s_∇ is a left A–map, then so is ∇, because $1 \otimes j$ is injective. From this, and its analogue on the other side, we deduce the following corollary.

Corollary 8.5. *Let A be an algebra and ${}_AE_A$ an A–bimodule. Then E has a right connection if and only if the natural map $E \otimes_k A \to E$ splits as a map of bimodules. Similarly, E has a left connection if and only if the natural map $A \otimes_k E \to E$ splits as a map of bimodules.*

Definition 8.6. *Let A be an algebra and E an A–bimodule. A* bimodule connection *on WE is a pair (∇_ℓ, ∇_r) where ∇_ℓ is a left connection on E and ∇_r is a right connection on E.*

Proposition 8.7. [CQ95a, Section 8] *Let A be an algebra and E an A–bimodule. There is a bimodule connection on A if and only if E is a projective object of $A -$* Bimod.

Proof. If E has a bimodule connection, then there exist bimodule maps $s_1 : E \to E \otimes_k A$ and $s_2 : E \to A \otimes_k E$ which split the respective multiplication maps. Then it is easy to check that $(s_2 \otimes 1_A)s_1 : E \to A \otimes_k E \otimes_k A$ splits the map $a \otimes \xi \otimes b \mapsto a\xi b$. Hence E is a summand of the free A–bimodule $A \otimes_k E \otimes_k A \cong (A \otimes_k A^{op}) \otimes_k E$, and so is a projective bimodule.

Conversely, if E is a projective A–bimodule, then there is a bimodule splitting $s : E \to A \otimes_k E \otimes_k A$. If $m_\ell : A \otimes_k E \to E$ and $m_r : E \otimes_k A \to E$ are the natural maps, then it follows from the above that $(\nabla_{(m_\ell \otimes 1)s}, \nabla_{(1 \otimes m_r)s})$ is a bimodule connection on E. $\square$

Recall that we showed that an algebra A is quasi-free if and only if $\Omega^1(A)$ is a projective bimodule if and only if $\Omega^1(A)$ has a right connection. (See Section 7.1.) In view of Proposition 8.7, $\Omega^1(A)$ should also have a left connection. The following proposition answers the question: what happened to the left connection?

Proposition 8.8. [CQ95a, Proposition 8.5] *Let A be an algebra. Then $\Omega^1(A)$ has a left connection if and only if it has a right connection.*

Proof. There is a bijection between left and right connections on $\Omega^1(A)$ given by sending a right connection ∇_r to $\nabla_\ell := \nabla_r + d$.

To show this, we identify $\Omega^1(A) \otimes_A \Omega^1(A)$ with $\Omega^2(A)$ via multiplication, as in Proposition 4.13. Then a right connection on $\Omega^1(A)$ may be viewed as a linear map $\nabla_r : \Omega^1(A) \to \Omega^2(A)$ which satisfies $\nabla_r(adbc) = a\nabla_r(db)c + adbdc$ for all $a, b, c \in A$. Setting $\nabla_\ell = \nabla_r + d$, we compute

$$\begin{aligned}
\nabla_\ell(adbc) &= a\nabla_r(db)c + adbdc + d(adbc) \\
&= a\nabla_r(db)c + adbdc + d(adb)c - adbdc \\
&= a(\nabla_r + d)(db)c + dadbc \\
&= a\nabla_\ell(db)c + dadbc
\end{aligned}$$

which shows that ∇_ℓ is a left connection. Similarly, if we start with a left connection ∇_ℓ on $\Omega^1(A)$, then $\nabla_r := \nabla_\ell - d$ is a right connection. □

Exercises 8.9. Exercises on connections.

(1) Show that if A is an algebra and ∇_r is a right connection on ${}_AA_A$ then A is separable.
(2) Show that, for any algebra A and any $n \geq 0$, there is a one-to-one correspondence between left connections on $\Omega^n(A)$ and right connections on $\Omega^n(A)$.

9. Going further

In this course, we did not cover all of the paper [CQ95a]. Some of the omitted sections are, in particular: Propositions 2.6 to 2.8 (relative forms on a tensor algebra); Proposition 2.9, Corollary 2.10 (cotangent exact sequence); Proposition 2.1 (uniqueness of separability element); Proposition 5.4 (quasi-freeness of a limit of quasi-free algebras); Section 6 (formal tubular neighbourhood theorem); Section 7 (universal liftings and separability elements); Section 8 (geodesic flow and the exponential map - this was briefly discussed in class).

Many of the results in [CQ95a] should not be considered in isolation. They were proved in order to be used in subsequent papers. Therefore, anybody who wishes to understand these results fully should look at the papers [CQ95b], [CQ95c], [CQ97]. A first step in studying these papers is to read about cyclic homology, which we were not able to cover due to lack of time. A reference for this is the book [Lod92]. In order to study cyclic homology, spectral sequences are essential. An excellent introduction to spectral sequences can be found in the lecture notes [Vak07].

Other sources of information about quasi-freeness are [Gin05, Section 19] and the references therein. There are 61 references to [CQ95a] listed on Mathscinet. It is worth looking at them to get an idea of how the notion of quasi-freeness is used in noncommutative algebra.

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