ALGEBRAIC GEOMETRY

Abstract. These notes were taken from a second course in algebraic geometry given by M. Stillman at Cornell University in autumn 2007. The rough notes taken in class were transcribed by R. Vale, who is responsible for any errors or spelling mistakes.

The reader is assumed to have read chapters I and II of the first volume of Shafarevich's "Basic Algebraic Geometry", 2nd edition.

1. Examples

Example. Let $L, M \subset \mathbb{P}^3$ be two disjoint projective lines. Let $\phi : \mathbb{P}^1 \to L$ and $\psi : \mathbb{P}^1 \to M$ be isomorphisms. Let

$$X = \bigcup_{t \in \mathbb{P}^1} \overline{\phi(t)\psi(t)} \subset \mathbb{P}^3.$$

Take the coordinates on $\mathbb{P}^3$ to be x, y, z, w. By making a projective change of coordinates, we may arrange that $L = V(z, w)$ and $M = V(x, y)$. We have

$$\phi : (s : t) \mapsto (s : t : 0 : 0)$$

$$\psi : (s : t) \mapsto (0 : 0 : s : t)$$

So the line $\overline{\phi(s:t)\psi(s:t)}$ is the set

$$\{(as : at : bs : bt) : (a : b) \in \mathbb{P}^1\}.$$

So $\mathbb{P}^1 \times \mathbb{P}^1 \to X \subset \mathbb{P}^3$. This is a surjection by definition of X. So X is the image of a morphism from $\mathbb{P}^1 \times \mathbb{P}^1$ and hence X is an algebraic set. Since X is the image of something irreducible, X is irreducible. Since X contains two distinct lines L and M, the dimension of X cannot be 0 or 1. Hence, X has dimension 2. But X is contained in $V(xw - yz)$ which is itself a closed irreducible subset of $\mathbb{P}^3$ of dimension 2. So $X = V(xw - yz)$, a *quadric*.

Example. Let $C \subset \mathbb{P}^m$, $D \subset \mathbb{P}^n$ be smooth curves and $\phi : C \to D$ an isomorphism. Let $\mathbb{P}^m$, $\mathbb{P}^n \subset \mathbb{P}^{m+n+1}$ be disjoint linear spaces (think of it as putting $\mathbb{P}^m$ in first lot of $m + 1$

coordinates and $\mathbb{P}^n$ in the last $n+1$ coordinates). Define

$$X = \bigcup_{p\in C} \overline{p\phi(p)} \subset \mathbb{P}^{m+n+1}.$$

If C, D are rational normal curves then X is called a *rational surface scroll.*

First question: is X algebraic? Let's take C to be a line.

$$C \times \mathbb{P}^1 \to \mathbb{P}^{m+n+1}$$

$$(p,(s:t)) \mapsto sp + t\phi(p)$$

is regular because ϕ is. If C is projective then $C \times \mathbb{P}^1$ is projective. So the image of $C \times \mathbb{P}^1$ under any regular map is closed, and therefore is algebraic.

Next question: what is $\dim(X)$? X is irreducible since it is the image of $C \times \mathbb{P}^1$, and so also it has dimension ≤ 2. But X contains C and a point not on C, so X (which is irreducible) must have dimension exactly 2. Note that we have shown that there is a regular map $C \times \mathbb{P}^1 \to X$. Projection onto the first $m+1$ coordinates is a rational map $X \dashrightarrow C$ which yields a rational map $X \dashrightarrow C \times \mathbb{P}^1$ which is inverse to $C \times \mathbb{P}^1 \to X$. Therefore X is birational to $C \times \mathbb{P}^1$.

Exercises.

(1) Check that the map $X \dashrightarrow C \times \mathbb{P}^1$ is a well-defined rational map which is inverse to $C \times \mathbb{P}^1 \to X$.
(2) Let D be a conic in $\mathbb{P}^2$ and $\mathbb{P}^1 = C \to D$ an isomorphism. Let $X = \bigcup_{p\in C} \overline{p\phi(p)} \subset \mathbb{P}^4$. Show that $X \cong \mathrm{Bl}_{\mathrm{pt.}}(\mathbb{P}^2)$.
(3) Suggested problems from Shafarevich III.1, numbers 1,4,12,18.

2. Weil divisors

Let X be an irreducible variety. Often X is assumed to be smooth, or sometimes we need only assume that X is smooth in codimension one, that is, $\mathrm{sing}(X)$ has codimension at least 2 in X.

Example. A basic example. On $\mathbb{A}^1$, let $f = \frac{(t+1)^2(t-2)}{(t+\frac{1}{2})^4(t-\pi)^3} \in k(\mathbb{A}^1)$. Then the *divisor* $\mathrm{div}(f)$ of f is (count poles and zeroes with multiplicities) $\mathrm{div}(f) = 2[-1] + [2] - 4[-\frac{1}{2}] - 3[\pi]$.

Definition. *A* prime divisor C *on* X *is an irreducible codimension one subvariety. A* divisor $D = k_1C_1 + k_2C_2 + \cdots + k_rC_r$ *is a formal sum of prime divisors* C_i *with* $k_i \in \mathbb{Z}$. D *is called* effective *if all the* k_i *are nonnegative and* $D \neq 0$. *We write this as* $D > 0$.

The support $\operatorname{supp}(D)$ *of* D *is* $\cup_{k_i \neq 0} C_i$.

$\operatorname{Div}(X)$ *is the free abelian group generated by the prime divisors.*

2.1. **Divisor of a function.** Let $f \in k(X)^* := k(X) \setminus \{0\}$. We can define

$$\operatorname{div}(f) = \sum_{\substack{C \subset X \\ C \text{ a prime divisor}}} \nu_C(f)C$$

where $\nu_C(f)$ is the order of the zero of f along C, or $-$(the order of the pole of f along C). Here is how to define $\nu_C(f)$. Assume X is smooth. By Chapter II of Shafarevich, we can choose some open affine $U \subset X$ so that $U \cap C \neq \varnothing$ and such that the ideal $I_{C \cap U}$ of C in U is generated by a single element $\pi \in k[U]$ (this is called choosing a local equation for C). If f is a regular function on U, ie. $f \in k[U]$, then there exists $\ell \geq 0$ such that $f \in (\pi^\ell)$ but $f \notin (\pi^{\ell+1})$. This uses the fact from commutative algebra that $\cap_{\ell \geq 1}(\pi^\ell) = 0$. Set $\nu_C(f) = \ell$. If $f = \frac{g}{h} \in k(U)$ with $g, h \in k[U]$ then set $\nu_C(f) = \nu_C(g) - \nu_C(h)$. Several things need to be checked (as an exercise):

(1) $\nu_C(f)$ doesn't depend on the choice of U or π.
(2) $\nu_C(g/h)$ doesn't depend on the particular choice of $f = g/h$.
(3) Given $f \in k(X)^*$, $\{C : \nu_C(f) \neq 0\}$ is finite.

Some properties of $\nu_C(f)$ are:

- $\nu_C(fg) = \nu_C(f) + \nu_C(g)$.
- $\nu_C(f + g) \geq \min\{\nu_C(f), \nu_C(g)\}$ with equality if $\nu_C(f) \neq \nu_C(g)$.

Some basic properties of $\operatorname{div}(f)$ are

(1) $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$.
(2) $\operatorname{div}(f) = 0$ if $f \in k^*$.
(3) $\operatorname{div}(f) \geq 0$ if $f \in k[X]$.

2.2. **Divisor class group.** Let $P(X)$ be the subgroup of $\operatorname{Div}(X)$ consisting of $\operatorname{div}(f)$ for $f \in k(X)^*$ (the subgroup of principal divisors).

Definition.

$$\text{Cl}(X) = \text{Div}(X)/P(X)$$

is the class group *of* X*.*

If $D, E \in \text{Div}(X)$, we write $D \sim E$ and say D and E are linearly equivalent, if $D - E = \text{div}(f)$ for some $f \in k(X)^*$.

2.3. **Simple properties.** Assume X is smooth and irreducible.

Proposition. $\text{div}(f) \geq 0 \implies f$ *is regular on* X*.*

Proof. Suppose f is not regular at $p \in X$. Write $f = g/h$ with $g, h \in \mathcal{O}_{X,p}$ but $g/h \notin \mathcal{O}_{X,p}$. Since X is smooth, $\mathcal{O}_{X,p}$ is a UFD, so we can choose g, h relatively prime. Let $\pi \mid h, \pi \nmid g$ and π a prime element of $\mathcal{O}_{X,p}$. There exists an open set $U \ni p$ such that $V(\pi) \cap U$ is irreducible of codimension one. Let $C = \overline{V(\pi) \cap U} \subset X$. Then $\nu_C(f) < 0$. Contradiction. So f is regular at p. □

It follows that if X is smooth and projective then $\text{div}(f) \geq 0$ implies f is constant. So $\text{div}(f) = \text{div}(g)$ implies $f = \alpha g$ for some $\alpha \in k^*$.

Examples.

(1) $X = \mathbb{A}^n$. Recall every codimension one subset of $\mathbb{A}^n$ is principal (ie. defined by a single equation). So $\text{Cl}(\mathbb{A}^n) = 0$.

(2) $X = \mathbb{P}^n$. Prime divisors are $V(F)$, $F \in k[X_0, \ldots, X_n]_d$ homogeneous of degree d and irreducible (this is a result from earlier in Shafarevich). If $f \in k(\mathbb{P}^n)^*$ then $f = \frac{F_1^{a_1} \cdots F_r^{a_r}}{G_1^{b_1} \cdots G_s^{b_s}}$ with $\sum a_i \text{deg} F_i = \sum b_j \text{deg} G_j$. Then $\text{div}(f) = \sum a_i V(F_i) - \sum b_j V(G_j)$. So

$$\text{Div}(X) \to \mathbb{Z} \to 0$$

$$D = \sum a_i C_i \mapsto \text{deg} D = \sum a_i \text{deg} C_i$$

has kernel exactly $P(X)$, and so $\text{Cl}(\mathbb{P}^n) = \mathbb{Z}$.

Exercises.

(1) X = rational scroll given by line and conic, $X \subset \mathbb{P}^4$. Compute $\mathrm{Cl}(X)$, using $X \cong \mathrm{Bl}_{\mathrm{pt.}}(\mathbb{P}^2)$.

(2) Show $\mathrm{Cl}(\mathbb{P}^m \times \mathbb{P}^n) = \mathbb{Z}^2$, or even for $m = n = 1$.

3. Cartier divisors

These are also called locally principal divisors. Suppose X is a smooth irreducible variety. If $D = \sum a_i C_i$ is a Weil divisor, $a_i \in \mathbb{Z}$, then by Shafarevich II.3, Theorem 1, C_i has a single local equation near every smooth point. That is, there exists an open affine cover such that if U is an open set in the cover then $C_i \cap U$ is defined by $\pi_i \in k[U]$ and

$$D|_U = \sum_{C_i \cap U \neq \varnothing} a_i C_i \cap U = \mathrm{div}(\pi_1^{a_1} \cdots \pi_r^{a_r}).$$

Therefore, on a cover $\{U_j\}$, we get $f_j \in k(X)^*$ such that $\mathrm{div}(f_j)|_{U_j} = D|_{U_j}$.

Definition. *Let X be an irreducible variety (doesn't need to be smooth!) A* Cartier divisor *or locally principal divisor on X is a system of data $\{(U_j, f_j)\}$ where*

- *$\{U_j\}$ is an open cover of X.*
- *$f_j \in k(X)^*$.*
- *f_i/f_j and f_j/f_i are regular on $U_i \cap U_j$ for all i, j.*

3.1. **Going back and forth.** Cartier $\rightsquigarrow$ Weil.

Given $\{(U_i, f_i)\}$ a Cartier divisor on X, let $D = \sum_{\substack{C \subset X \\ \text{prime divisors}}} k_C C$ where if $C \cap U_i \neq \varnothing$, set $k_C = \nu_C(f_i)$. Check: if $U_j \cap C \neq \varnothing$ then $\nu_C(f_j) = \nu_C(f_i \cdot \frac{f_j}{f_i}) = \nu_C(f_i) + \nu_C(f_j/f_i)$. But $\nu_C(f_j/f_i) = 0$ since f_i/f_j is regular and has no poles on the dense open set $U_i \cap U_j$. Check as exercise: for a finite cover $\{U_i\}$, only finitely many of the k_C can be nonzero.

Weil $\rightsquigarrow$ Cartier.

Assume X is smooth. We did this above in the definition of Cartier divisor.

Example. $X = V(y^2 - x^3) \subset \mathbb{A}^2$. $p = (0, 0)$ *is a prime divisor but not locally principal, loosely speaking because* "$\mathrm{div}(x) = 2p$", "$\mathrm{div}(y) = 3p$".

In the definition of Cartier divisor, $\{(U_i, f_i)\}$ and $\{(V_j, g_j)\}$ are considered the same if f_i/g_j and g_j/f_i are regular on $U_i \cap V_j$ for all i, j.

Group structure: if $D = \{(U_i, f_i)\}$ and $E = \{(V_j, g_j)\}$ then $D + E = \{(U_i \cap V_j, f_i g_j)\}$. Check that this matches up with the addition for Weil divisors in the situation where these Cartier divisors correspond to Weil divisors.

Writing $\mathrm{CaDiv}(X)$ for the group of Cartier divisors, we get

$$\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$$

is an isomorphism of abelian groups for X smooth and irreducible.

Definition. *A principal Cartier divisor is* $\mathrm{div}(f) = \{(X, f)\}$ *for* $f \in k(X)^*$. *The subgroup of* $\mathrm{CaDiv}(X)$ *consisting of the principal Cartier divisors is denoted* $\mathrm{Ca}P(X)$.

Under the above isomorphism, $\mathrm{Ca}P(X)$ is identified with $P(X)$.

Definition. *The* Picard group

$$\mathrm{Pic}(X) = \mathrm{CaDiv}(X)/\mathrm{Ca}P(X)$$

.

Theorem. *If X is smooth and irreducible then* $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$.

Example. If $X \subset \mathbb{P}^n$ is smooth and irreducible, let $F \in k[x_0, \ldots, x_n]$ be a homogeneous form of degree D which does not vanish identically on X. We may associate a Cartier divisor to F.

Definition.

$$\mathrm{div}(F) = \{(U_i := X \setminus V(x_i), F/x_i^d)\}$$

is the Cartier divisor associated to F.

This is really a Cartier divisor because $(x_i/x_j)^d$ is regular on $U_i \cap U_j$ for all i, j. What's the corresponding Weil divisor? Factor F in $k[x_0, \ldots, x_n]$ as $F = F_1^{a_1} \cdots F_r^{a_r}$ with $a_i \geq 1$, F_i irreducible. Then $\mathrm{div}(F) = \sum_{i=1}^r a_i V(F_i)$ as a Weil divisor.

3.2. **Support of a Cartier divisor.** If $D = \{(U_i, f_i)\}$ is a Cartier divisor then

$$\mathrm{supp}(D) = \{p \in X : \text{ if } p \in U_i \text{ then } f_i(p) = 0 \text{ or } f_i \text{ is not regular at } p\}.$$

Exercise. *Let $Q \subset \mathbb{P}^3$ be $Q = V(xw - yz)$. Find the divisor of y/x as a Weil and as a Cartier divisor. Do the same for x and for y.*

4. Solutions to selected exercises

(1) $p = (1:0:0)$, $\mathrm{Bl}_p(\mathbb{P}^2) = V\left(\begin{vmatrix} x_0 & x_1 & x_2 \\ x_3 & x_2 & x_5 \end{vmatrix}\right) \subset \mathbb{P}^4$ (via Segre embedding). Take a conic C to be $V(xy - z^2) \subset \mathbb{P}^2$. Isomorphism $\phi : \mathbb{P}^1_{ab} \to C$ is given by $(a:b) \mapsto (a^2 : b^2 : ab)$. Embed these in $\mathbb{P}^4$ as $(a:b:0:0:0)$ and $C = (0:0:x:y:z)$. Then the line $\overline{p\phi(p)}$ is $(sa : sb : ta^2 : tb^2 : tab)$, and X = set of all such points in $\mathbb{P}^4$. The points $(y_0 : y_1 : y_2 : y_3 : y_4)$ of X satisfy the equations $y_0y_3 = y_1y_4$, $y_1y_2 = y_0y_4$, $y_2y_3 = y_4^2$. If we identify $y_4 = x_2$, $y_2 = x_1$, $y_3 = x_5$, $y_0 = x_0$ and $y_1 = x_3$ then we see that this is isomorphic to $\mathrm{Bl}_p(\mathbb{P}^2)$ as was claimed. Under the Segre embedding of $\mathrm{Bl}_p(\mathbb{P}^2)$ into $\mathbb{P}^5$, a point $([z_0 : z_1 : z_2], [p:q])$ is mapped to $(z_0p : z_1p : z_2p : z_0q : z_1q : z_2q)$ which we identify with $(z_0p : z_1p : z_2p : z_0q : z_2q) \in \mathbb{P}^4$ since the equations for the blowup say $z_2p = z_1q$. Since the exceptional divisor E is given by $z_1 = z_2 = 0$, we see that E corresponds to $V(x_1, x_2, x_5)$ as a subset of X.

(2) To compute $\mathrm{div}(x)$ on $X = V(xy = zw) \subset \mathbb{P}^1$. On $U_x = \{x \neq 0\}$ we have $x/x = 1$ so divisor is just $(U_x, 1)$ or 0 as a Weil divisor. On U_y, we must consider x/y. The ring $k[U_y] = k[x,z,w]/(x = zw) = k[z,w]$, so the divisor of x/y here is $L + M$ where $L = V(z,x)$ with local equation z and $M = V(w,x)$ with local equation w. For every prime divisor C on X, either $C \cap U_x \neq \varnothing$ or $C \cap U_y \neq \varnothing$. So we have computed enough, and $\mathrm{div}(x) = L + M$.

5. Pullback of a divisor class

The following useful proposition is in Hartshorne but not in Shafarevich.

Proposition. *Let X be a smooth variety, $U \subset X$ dense and open, $Z = X \setminus U$.*

(1) *If* $\mathrm{codim}_X Z \geq 2$ *then* $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$.

(2) *If $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r \cup W$ where the Z_i are prime divisors and* $\mathrm{codim}_X W \geq 2$, *then there exists an exact sequence*

$$\mathbb{Z}^r \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

where $e_i \in \mathbb{Z}^r \mapsto [Z_i]$ and $D \in \mathrm{Cl}(X) \mapsto D|_U$. The first map may not be injective.

Proof. We check that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is well-defined. If $D \sim E$ on X then $D - E = \mathrm{div}(f)$ for $f \in k(X)^* = k(U)^*$. So $D|_U - E|_U = \mathrm{div}(f)|_U = \mathrm{div}_U(f)$, ie. the divisor of f regarded as a rational function on U. So we do have a map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$. This map is surjective because any prime divisor $C \in \mathrm{Div}(U)$ is the image of the closure $\overline{C} \in \mathrm{Div}(X)$. We only need to show that the kernel of this map consists of linear combinations of the Z_i. If $D = \sum n_i \overline{C_i} + \sum m_i Z_i$ is in the kernel, then $\sum n_i C_i = 0$. So there is an $f \in k(U)^*$ with $\mathrm{div}(f) = \sum n_i C_i$. But then $f \in k(X)^*$ and $\mathrm{div}(f)$ is $\sum n_i \overline{C_i}+$ (some combination of the Z_i). So D is also a combination of the Z_i as required. □

5.1. **Relationship of divisors and regular maps.** Let $\phi : X \to Y$ be regular where X, Y are smooth irreducible varieties. Let $D \in \mathrm{Div}(Y)$. We want to define a pullback or inverse image $\phi^* D$ of D.

Example.

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\sim} & Q \\ & \searrow & \downarrow \\ & & \mathbb{P}^3 \end{array}$$

where $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a quadric in $\mathbb{P}^3$. Then $Q \subset \mathbb{P}^3$ is a prime divisor. $H = V(x)$ is another prime divisor. What should $\phi^* Q$ and $\phi^* H$ be? Basically, $\phi^* H$ should be the inverse image of $x = 0$; a line. But $\phi^{-1} Q = \mathbb{P}^1 \times \mathbb{P}^1$, which is not a divisor.

Let $D = \{(U_i, f_i)\}$, a Cartier divisor on Y, $f_i \in k(Y)^*$. Suppose $\phi(X) \nsubseteq \mathrm{supp}(D)$. Let $\phi^* D = \{(\phi^{-1} U_i, \phi^* f_i)\}$. Check that $\phi^* D$ is a Cartier divisor on X (indeed, $\{\phi^{-1} U_i\}$ is a cover of X and $\phi^* f_i \in k(X)^*$ because $\phi(X) \nsubseteq \mathrm{supp}(D)$.)

If $D, E \in \mathrm{Div}(Y)$ and $\phi(X) \nsubseteq \mathrm{supp}(D) \cup \mathrm{supp}(E)$ then $\phi^*(D + E) = \phi^*(D) + \phi^*(E)$.

If ϕ is surjective, or even dominant (meaning $\phi(X)$ dense in Y), then $\phi(X)$ can never be contained in $\mathrm{supp}(D)$ and so $\phi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ is well-defined and induces $\phi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$.

Examples. Suppose X, Y smooth, $\phi : X \to Y$ regular, $D \subset Y$ prime (eg. X, Y curves, ϕ finite, D pt.) What is $\mathrm{supp}(\phi^* D)$? Looking at the local picture, choose $U \subset Y$ affine open such that $V = \phi^{-1} U$ is affine and with $D|_U = (\pi) \subset k[U]$ (note to self: why can you choose a U like this?)

ϕ is pullback of functions. $\phi^*(D)$ defined by $\phi^*\pi = \pi \circ \phi$ on V. Can't have poles on V (it's a regular function). Zeroes; $(\pi \circ \phi)(q) = 0$ implies $\pi(\phi(q)) = 0$, $q \in V$ ie. $\phi(q) \in D|_U$. So $q \in \phi^{-1}(D|_U)$. So $\mathrm{supp}(\phi^* D) = \phi^{-1}(D)$ (D prime).

6. Moving the support of a divisor

Last time: $\phi : X \to Y$ regular map, X, Y smooth. $D \in \mathrm{Div}(Y)$, $\phi(X) \not\subseteq \mathrm{supp}(D)$. Can define $\phi^* D \in \mathrm{Div}(X)$.

Example. Suppose $\phi \in k(X)^*$. Then ϕ defines a function $\phi : X \to \mathbb{P}^1$, a regular map if X is a smooth curve, since the codimension of the set of points where ϕ fails to be regular is ≥ 2. Exercise: $\phi^*(0) - \phi^*(\infty) = \phi^*(0 - \infty) = \mathrm{div}(\phi)$. This is OK if ϕ is not constant, because $\phi(X)$ is either a point or all of $\mathbb{P}^1$.

Example.

$$i : C \hookrightarrow \mathbb{P}^2 \supset D$$

C, D smooth irreducible curves, $C \neq D$, eg. D line.

Exercises.

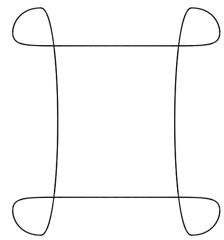
1. *What is $i^* D \in \mathrm{Div}(C)$?*
2. *What about if $D = C$? (we're about to define it).*

Theorem 1 (Moving the support of a divisor)**.** *For any divisor D on a smooth X and any finite number of points $P_1, \ldots, P_m \in X$, there exists $D' \sim D$ such that $P_1, \ldots P_m \notin \mathrm{supp} D'$.*

Corollary. *Let $\phi : X \to Y$ be a regular map of smooth irreducible varieties. Then there is an induced group homomorphism $\alpha = \phi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$.*

Proof. Construction/proof for corollary: If $D \in \mathrm{Div}(Y)$ and $\phi(X) \not\subseteq \mathrm{supp}(D)$ then $\phi^* D$ has been defined. Take $\alpha = \phi^*$ on the class group. If $D \in \mathrm{Div}(Y)$ and $\phi(X) \subset \mathrm{supp}(D)$ we can choose $D' \sim D$ such that $\phi(X) \not\subseteq \mathrm{supp}(D')$ (take $x \in \phi(X)$ and $D' \sim D$ with $x \notin \mathrm{supp}(D')$.) Then define $\alpha([D]) = [\phi^*(D')]$. Check as an exercise that this α is well-defined. $\square$

Example. Let X = smooth quartic curve in $\mathbb{P}^2$. Real picture might look like:



$D = p + q$. Find $D' \sim D$ which misses p and q.

First try: can we write $p + q \sim p' + q'$ for some p', $q' \in X$ not p, q? No! (cf. if $D = p$ and $D \sim p'$ then $\exists \phi : X \to \mathbb{P}^1$ with one zero and one pole. Therefore has degree 1, therefore ϕ iso. But $X \not\cong \mathbb{P}^1$).

The problem can be simplified to: given $D = p$, find $D' \sim p$ but missing p, q (if so, get also $D'' \sim q$ missing p, q and so $p + q \sim D' + D''$ as claimed). Want $p + \mathrm{div}(\phi)$ to miss p, q. Let ℓ be the equation of a line through p which doesn't go through q and is not tangent to p (you can find one since X is smooth). Let m = equation of a line in $\mathbb{P}^2$ missing p and q. Let $\phi = m/\ell$. Then

$$\mathrm{div}(\phi) = \underbrace{r_1 + r_2 + r_3 + r_4}_{\text{not } p \text{ or } q} - p - \underbrace{s_1 - s_2 - s_3}_{\text{not } p \text{ or } q}$$

So $p \sim p + \mathrm{div}(\phi) \leftarrow$ support doesn't contain p or q.

Exercises.

(1) *Suppose $\phi : X \dashrightarrow Y \subset \mathbb{P}^n$ is a rational map, X, Y smooth. Show there exists a homomorphism $\phi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ (idea: set of points where ϕ is not regular, called* base locus*, is of codimension at least 2).*

(2) $X = \mathrm{Bl}_{\text{pt.}}(\mathbb{P}^2)$ *and $E \subset X$ exceptional locus.* $\mathbb{P}^1 \cong E \overset{i}{\hookrightarrow} X \supset E$.
$i^*(E) \in \mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$. *What is it?*

6.1. **Proof of theorem 1.** WLOG assume D is a prime divisor, X affine (to do this, take a hyperplane not containing any of the points p_i and intersect its complement with X). Assume by induction that $p_1, \ldots, p_{m-1} \notin \mathrm{supp}(D)$ but $p_m \in \mathrm{supp}(D)$. Find $D' \sim D$ such that $p_1, \ldots, p_{m-1}, p_m \notin \mathrm{supp}(D')$. Our plan: find a local equation of D missing $p_1, \ldots, p_m$ in $k[X]$.

First, let $\pi_1 \in \mathcal{O}_{X,p_m}$ be a local equation for D in a nbd of p_m. Now let $\pi_2 \in k[X]$ be a local equation for D in a nbd of p_m. This can be accomplished by clearing denominators as follows. The divisor of poles $\operatorname{div}_\infty(\pi_1)$ is $\sum k_\ell F_\ell$ with $p_m \notin F_\ell$ since π_1 is regular at p_m. Let $f_\ell \in k[X]$ be a function which vanishes on F_ℓ but does not vanish at p_m. Since we assume X is affine, there exists such a function on the ambient $\mathbb{A}^n$, and we can just restrict it to X. Take $\pi_2 = \pi_1 \prod f_\ell^{k_\ell}$. (Continued in next lecture).

7. Riemann-Roch spaces

7.1. **Rest of proof of Theorem 1.** π_2 is a local equation of D in a nbd of p_m since $f_\ell \in \mathcal{O}_{X,p_m}$ for all ℓ. Each $f_\ell^{k_\ell}$ cancels at least a $k_\ell F_\ell$ from (π_1), so π_2 is regular on X.

Now we choose $\pi \in k[X]$ a local equation for D near p_m such that $\pi(p_i) \neq 0$, $1 \leq i \leq m-1$. This can be done as follows. For $1 \leq i \leq m-1$, let $g_i \in k[X]$ vanish on D and on $p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m$, $g_i(p_i) \neq 0$. Let

$$\pi = \pi_2 + \sum_{i=1}^{m-1} \alpha_i g_i^2, \qquad \alpha_i \in k$$

be such that $\pi(p_i) \neq 0$, $1 \leq i \leq m-1$. To do this, choose α_i such that $\pi_2(p_i) + \alpha_i (g_i(p_i))^2 \neq 0$. Note that in $\mathcal{O}_{X,p_m}$, $\pi = \pi_2(1 + \pi_2 \sum \alpha_i \beta_i^2)$ where $g_i = \beta_i \pi_2$ for some $\beta_i \in \mathcal{O}_{X,p_m}$. Since $1 + \pi_2 \sum \alpha_i \beta_i^2$ is a unit in $\mathcal{O}_{X,p_m}$, $\pi \in k[X]$ is a local equation for D near p_m such that $\pi(p_i) \neq 0$, $i = 1, 2, \ldots, m-1$. Finally, $D' = D - \operatorname{div}(\pi) = -\sum r_s D_s$, $r_s > 0$ (as a divisor on X - now we are including the non-affine part which was deleted at the beginning of the proof). So $\operatorname{div}(\pi) = D + \sum r_s D_s$. Note $p_1, \ldots, p_{m-1} \notin \operatorname{supp}(D')$. What about p_m? $p_m \notin D_s$ for any s by construction, since $\operatorname{div}(\pi)|_U = D$ so $D_s \cap U = \varnothing$ for all s for some $U \ni p_m$. So $p_m \notin \operatorname{supp}(D') = D - \operatorname{div}(\pi)$.□

7.2. **Linear systems.**

Example. $\mathbb{P}^1_{xy}$, $k(\mathbb{P}^1) = k(t)$, $t = y/x$, $p_\infty = (0:1) \in \mathbb{P}^1$. Then $\deg f(t) \leq n$ if and only if $\operatorname{div}(f) \geq -n \cdot p_\infty$, ie. $\operatorname{div}(f) + np_\infty \geq 0$. (Because

$$\left(\frac{y}{x}\right)^n + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + \cdots + a_0$$

has a pole of order n at ∞).

Definition. *As usual let X be smooth, $D \in \operatorname{Div}(X)$. Define*

$$L(D) = \{f \in k(X)^* : \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

the Riemann-Roch space*, also called* $H^0(D)$*.*

Remarks.

(1) $L(D)$ is a vector space over k.
(2) Theorem: If X is projective then $\dim L(D) < \infty$ for all D.
(3) Define $\ell(D) := \dim_k L(D)$.

Example. $X =$ curve, $p, q, r \in X$. Then

$$L(p + 2q - r) = \{f \in k(X)^* : f \text{ has at most a pole of order one at each of } p, q, \\ f \text{ does not vanish at } r. \text{ No other poles, but maybe other zeroes.}\}$$

Example. On $\mathbb{P}^1$, $L(np_\infty) = \{f = \frac{g(x,y)}{x^n}$ with g homog of degree $n\}$. $L(0) = k$ (X projective). $L(-C) = 0$, C prime divisor.

Exercises.

(1) *Let* $E \subset \mathbb{P}^2$ *be a smooth plane cubic (ie. choose one). Choose* $p \in E$*. Find* $L(p)$*,* $L(2p)$ *and* $L(3p)$*.*
(2) *Let* $X = V(x^3y + y^3z + z^3x) \subset \mathbb{P}^2$ *be the Klein quartic. Show:*
   (a) X *is smooth.*
   (b) $p = (0, 0, 1)$*. Fine* $L(p)$*,* $L(2p)$*,* $L(3p)$*.*
(3) *Pencil of conics example from Shafarevich.*

8. Linear systems

Recall: X smooth, $L(D) = \{f \in k(X) : D + \operatorname{div}(f) \geq 0\} \cup \{0\}$, a k–vector space.

Theorem. *If* X *is projective then* $\dim L(D) = \ell(D) = h^0(D) < \infty$*.*

Proposition. *If* $D \sim E$ *then* $L(D) \cong L(E)$*.*

Proof. If $D - E = \operatorname{div}(g)$ then

$$L(D) \to L(E)$$

$$f \mapsto fg$$

and $\operatorname{div}(f) + D \geq 0$ implies $\operatorname{div}(f) + \operatorname{div}(g) + E = \operatorname{div}(f) + D = \operatorname{div}(fg) + E \geq 0$. $\square$

Definition. *Let $W \subset L(D)$ be a vector subspace. Then $\{D + \operatorname{div}(f) : f \in W \setminus 0\}$ is called a* linear system*.*

$$|D| := \{D + \operatorname{div}(f) : f \in L(D)^*\}$$

is called a complete linear system/series*. This is the set of all effective divisors linearly equivalent to D.*

Examples. (1) $X = \mathbb{P}^1$, $p \in X$. Then $|p| = \{D + \operatorname{div}(f) : f \in L(p)^*\} = \{D : D \sim p \text{ and } D \text{ effective.}\}$. D has to have degree 1 (not proved yet) and so $|p| = \{q \in \mathbb{P}^1 : q \sim p\} = \mathbb{P}^1$.

(2) If $X = E \subset \mathbb{P}^2$ is a smooth cubic, it turns out that $p \sim D$, D effective, implies $D = p$. So $|p| = \{p\}$.

(3) $X = \mathbb{P}^2$, $L =$ line. Then $|L| =$ set of all lines in $\mathbb{P}^2$. If M is a line then the divisor of the rational function m/ℓ is $M - L$, where m, ℓ are the equations of M, L respectively. You can't get anything else because if $\operatorname{div}(f) = D - L$ with D effective then f has poles of order one exactly along L, so the denominator of f is forced to be ℓ. But then the numerator must have degree 1.

Let $p \in \mathbb{P}^2$. Then $|L|(-p) := \{M \in |L| : p \in M\}$ is a linear system. Here $W = \{f \in k(X)^* : f \in L(L), f(p) = 0\} \cup \{0\} = \{\frac{m}{\ell} : m(p) = 0\}$ (check as exercise).

Remark. Suppose $L(D) = \operatorname{span}(f_0, \ldots, f_r)$. Then $|D| \cong \mathbb{P}^r$. In fact, $|D| = \mathbb{P}(L(D) \setminus \{0\})$.

Note: if $D \sim E$ then $|D| = |E|$.

If $W \subset L(D)$ is a vector space then $\mathbb{P}W \subset D$.

Let $W \subset L(D)$ and $V \subset D$ be the corresponding linear system.

Definition. *The* base locus *of V is*

$$\bigcap_{E \in V} \operatorname{supp}(E).$$

If C is a prime divisor and $C \subset$ base locus of E, then C is called a *base component* of E. If the base locus of V is $\varnothing$, then V is called *basepoint-free*.

Example. $|L|$ in $\mathbb{P}^2$ is basepoint-free, since there are three lines with $L_1 \cap L_2 \cap L_3 = \varnothing$.

Example. $|L|(-p)$ has base locus p.

Example. Let $E \subset \mathbb{P}^2$ be a smooth cubic. Let $p \in E$, let $V =$ lines through p, that is, $i : E \hookrightarrow \mathbb{P}^2$. Let $V = \{i^*L : L \subset \mathbb{P}^2 \text{ line with } p \in L\}$. p is a base component because $i^*L = p + x + y$ for any L, where x, y are some points on E (this is a consequence of Bézout's theorem - see later). Then

$$\{i^*L - p : p \in L\} \subset |i^*L_0 - p|$$

where L_0 is some choice of line. Check as an exercise that this is actually an equality.

8.1. **Construction of rational maps.** Let X be projective and $W \subset L(D)$ be a subspace and $V \subset |D|$ be the corresponding linear system. Let $f_0, \dots, f_r$ be a basis for W. Define

$$\varphi_W : X \dashrightarrow \mathbb{P}^r$$
$$x \mapsto (f_0(x) : f_1(x) : \cdots : f_r(x))$$

If V has no base components then the domain of definition of φ is $X\setminus$ base locus. If V has base components, it can be larger.

Exercises.

(1) *If V is basepoint-free, then φ_V is regular.*
(2) *If $H \in \mathrm{Div}(\mathbb{P}^r)$ is a hyperplane then find $\varphi_V^*(H)$ and show $\varphi_V^*(H) \sim D$, assuming V has no base components.*
(3) *If $\varphi : X \dashrightarrow \mathbb{P}^r$ is any rational map and $D = \varphi^*(H)$, then there is $W \subset L(D)$ such that $\varphi = \varphi_W$.*

Exercise. $X = \mathbb{P}^1$, $p = (0 : 1)$. *Find $\phi_{|3p|} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^r$. What is r?*

9. Divisors on curves

Let X be a projective smooth curve. If $D \in \mathrm{Div}(X)$, $D = \sum_{i=1}^r n_i p_i$, $p_i \in X$. Define $\deg(D) = \sum n_i$.

Theorem. *If $f : X \to Y$ is a regular map of smooth irreducible projective curves (ie. a rational map) such that $f(X) = Y$ (ie. f is not constant) then* $\deg(f) = \deg f^*(q)$ *for all* $q \in Y$.

Corollary. $\deg(\operatorname{div}(g)) = 0$ *for all* $g \in k(X)^*$.

Proof. $g : X \to \mathbb{P}^1$. If $g \in k^*$ then $\operatorname{div}(g) = 0$ so done. If $g \notin k^*$, then g is a surjective map to $\mathbb{P}^1$. Then $\operatorname{div}(g) = g^*(0) - g^*(\infty)$ so $\deg(\operatorname{div}(g)) = 0$. □

Definition. $\mathrm{Cl}^0(X) = \mathrm{Pic}^0(X) = \{D \in \mathrm{Cl}(X) : \deg(D) = 0\}$ *(well-defined since* $D \sim E \implies \deg(D) = \deg(E)$*).*

Corollary. *X smooth irred projective curve. Then $X \cong \mathbb{P}^1$ iff* $\mathrm{Cl}^0(X) = 0$ *iff every degree zero divisor is principal.*

Proof. ($\implies$) holds since $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$. Conversely, if $\mathrm{Cl}^0(X) = 0$ then for any $p \neq q$, $p - q \in \mathrm{Cl}^0(X)$, so there is a $g \in k(X)^*$ such that $g : X \to \mathbb{P}^1$ has degree 1. So $[k(X) : k(\mathbb{P}^1)] = 1$ and $k(X) \cong k(\mathbb{P}^1)$ (via g). Therefore, $X \cong \mathbb{P}^1$. □

Theorem. *Let X be a smooth projective irreducible curve, $D \in \mathrm{Div}(X), L(D) \neq 0$. Then*

(1) $\dim_k L(D) \leq \deg(D) + 1$.
(2) *If $X \ncong \mathbb{P}^1$ then* $\dim_k L(D) \leq \deg(D)$.

In particular, $\dim_k L(D) < \infty$ *for all* $D \in \mathrm{Div}(X)$.

Proof. WLOG we may assume D is effective. $L(D) = \{f \in k(X)^* : D + \operatorname{div}(f) \geq 0\} \cup \{0\}$. Therefore there exists $f \in L(D)$ with $f \neq 0$ and $E + \operatorname{div}(f) \geq 0$, $E \sim D$ so $L(E) \cong L(D)$.

$\dim L(0) = 1$.

$\dim L(p) = ?$.

If $f \in L(p) \setminus L(0)$ then $\operatorname{div}(f) = q - p$, $q \neq p$. So by the above theorem, $X \cong \mathbb{P}^1$. So if $\dim L(p) \geq 2$ then $\dim L(p) = 2$ by direct calculation for $\mathbb{P}^1$. Now suppose $D = p_1 + p_2 + \cdots + p_d$ where $d = \deg(D)$ and some of the p_i can be the same.

$$L(0) \subset L(p_1) \subset L(p_1 + p_2) \subset \cdots L(p_1 + p_2 + \cdots + p_d) = L(D)$$

Claim: $\dim(L(E + p)/L(E)) \leq 1$. If so, $\dim L(D) \leq \deg(D) + 1$ and if $X \ncong \mathbb{P}^1$ then $\dim L(D) \leq \deg(D)$. To prove the claim:

Suppose $E = (n-1)p + E'$, $p \notin \text{supp}(E')$, $n \geq 1$. Let $\pi \in \mathcal{O}_{X,p}$ be a local equation for $p \in X$. Suppose $f, g \in L(E+p) \setminus L(E)$. Then f, g have poles of order $\leq n$ at p so $f = \alpha/\pi^n$, $g = \beta/\pi^n$ where $\alpha, \beta \in \mathcal{O}_{X,p}$ and $\alpha(p), \beta(p) \neq 0$. Then

$$\beta(p)f - \alpha(p)g = \frac{\alpha\beta(p) - \beta\alpha(p)}{\pi^n}.$$

The denominator has a factor of π because it is zero at p. So $\beta(p)f - \alpha(p)g \in L(E)$. This proves the claim. $\square$

9.1. **Rational maps and linear systems.** X projective smooth irreducible variety. $f = (f_0 : f_1 : \cdots : f_r) : X \dashrightarrow \mathbb{P}^r$. f gives a linear system $W \subset L(D)$. Find a D such that $f_i \in L(D)$ for all i, so $(f_i) + D \geq 0$. $\text{div}(f_i) = \sum_{j=1}^{s} n_{ij} C_j$ with C_j prime divisors, and some n_{ij} allowed to be zero. Let $k = \min_i\{n_{ij}\}$, $D = \sum_j -k_j C_j$. Then $f_i \in L(D)$ for all i since $n_{ij} - k_j \geq 0$ for all j and all i. Then $W = kf_0 + \cdots + kf_r$ is a subspace of $L(D)$ such that f is φ_W.

(ie. every rational map to $\mathbb{P}^r$ comes from some linear system).

10. Solutions to selected exercises

(1) $E : Y^2Z = X^3 + Z^3$, $p = (0:1:0)$. The exercise is to find $L(p), L(2p)$ and $L(3p)$.

U_X, U_Y, U_Z open sets. On U_X, $y^2z = 1 + z^3$. Can ignore this set since $z = 0$ implies $y \neq 0$, so $U_Y \cup U_Z$ cover.

$$U_Y : z = x^3 + z^3$$

$$U_Z : y^2 = x^3 + 1$$

$L(p) \supset k$. Functions on $y^2 = x^3 + 1$ that can have a pole at ∞ are x, y, $x + ay$. Look at x/z, y/z (in order to be able to compute something). On U_y, consider $\text{div}(x/z)$. Local equation at $(0,0)$ is $x = 0$ because $z(1+z)(1-z) = x^3$ so $z = \frac{1}{1-z^2}x^3 \in (x) \subset \mathcal{O}_{X,p}$. Alternatively, you could argue that a local equation is given by an element of $m \setminus m^2$. But $z \in m^2$ so x must be a local equation. $\frac{x}{z} = (1-z^2)\frac{x}{x^3} = (\text{unit}) \times x^{-2}$. So $\nu_p(x/z) = -2$. Same calculation yields $\nu_p(y/z) = \nu_p(1/z) = -3$. $\frac{x}{z}$ can't have other poles because it is regular on U_z. You get

$$L(p) = k$$
$$L(2p) = k + k\frac{x}{z}$$
$$L(3p) = k + k\frac{x}{z} + k\frac{y}{z}$$

where $L(p) = k$ either because X is not rational, or (better) because $x + ay$ can never have a pole of order 1 at p, since x has a pole of order 2 and y has a pole of order 3 there.

(2) $\mathbb{P}^2_{xyz} \times \mathbb{P}^1_{st} \xrightarrow{\sigma} \mathbb{P}^2$. $X = V(yt - zs)$. Then $\sigma|_X : X \to \mathbb{P}^2$ is $\mathrm{Bl}_{(1,0,0)}\mathbb{P}^2$. $E \subset X$ is $V(y, z)$, the exceptional divisor of the blowup. $i : E \hookrightarrow X$ gives a map $i^* : \mathrm{Cl}(X) \to \mathrm{Cl}(E)$. The exercise is to find $i^*(E)$.

Need $D \in \mathrm{Div}(X)$ such that $D \sim E$ and $E \not\subseteq \mathrm{supp}(D)$. Let $U = U_x \times U_s$, ie. $xs \neq 0$ ie. $U = X \setminus V(xs)$. On U, $E = V(y)$. Consider x/y. Then $\mathrm{div}(x/y)|_U = -E$ since $k[U] = \frac{k[y,z,t]}{(yt-z)} \cong k[y, t]$. $x/y = x \cdot y^{-1}$ and y is a local equation for E on U.

Now we need to look at $X \setminus U = V(xs, yt - zs) = V(x, yt - zs) \cup V(y, s)$. Let $C_1 = V(x, yt - zs)$ and $C_2 = V(y, s)$. Need to compute $\nu_{C_1}(x/y)$ and $\nu_{C_2}(x/y)$.

Take $U' = U_z \times U_t$, $U' \cap C_1 \neq \varnothing$, $U' \cap C_2 \neq \varnothing$. On U', $C_2 = V(y)$ and $C_1 = V(x)$. So $\mathrm{div}(x/y)|_{U'} = C_1 - C_2$ and so overall

$$\mathrm{div}(x/y) = C_1 - C_2 - E$$

(note that it doesn't have degree zero! It doesn't have to, since this is not a curve!).

Now calculate $i^*(C_1 - C_2) = i^*(E)$. The claim is that this is $-q$ for some $q \in E$. $C_1 \cap E = V(x, y, z) = \varnothing$ so need only consider $C_2 \cap E$. As a Cartier divisor, $C_2 = \{(U', y/z), (U, 1), \ldots\}$.

$i^*(y/z)$ on E is s/t. $C_2 \cap E = (1 : 0 : 0) \times (0 : 1) = p \times (0 : 1) =: q$. So $i^*(C_1 - C_2) = -q$. $E \cong \mathbb{P}^1$ so $D \sim (\deg(D)) \cdot$ (a generator) for any D. So $i^*(E) = -1$ as an element of $\mathrm{Cl}(\mathbb{P}^1)$.

11. Divisors on curves, part II

Theorem. *$f : X \to Y$ surjective regular map where X and Y are smooth projective curves. Then $\deg(f) = \deg f^*(q)$ for all $q \in Y$.*

Working on proof: (note that f is a finite map by Shafarevich II, 5.3, theorem 8, so $\deg(f)$ is well-defined). More notation: $f^* : k(Y) \to k(X)$ is an inclusion of fields. We will regard

$k(Y) \subset k(X)$. Degree of the extension is $[k(X) : k(Y)] = \deg(f)$. Let $\{p_1, \ldots, p_r\} = f^{-1}(q)$. Let $\tilde{\mathcal{O}} = \cap_{i=1}^r \mathcal{O}_{X,p_i}$, ie. those rational functions which are regular at each of the p_i.

$$\begin{array}{ccc} k(Y) & \subset & k(X) \\ \cup & & \cup \\ \mathcal{O}_{Y,q} & \subset & \tilde{\mathcal{O}} \end{array}$$

[Once we do sheaves, we really have $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and $\mathcal{O}_{Y,q} \to (f_*\mathcal{O}_X)_q = \tilde{\mathcal{O}}$].

Lemma 1. *$\tilde{\mathcal{O}}$ is a PID, with exactly r prime ideals (so in particular is a semilocal ring). There exist $t_1, \ldots, t_r \in \tilde{\mathcal{O}}$ such that $\nu_{p_i}(t_j) = \delta_{ij}$ for $1 \le i, j \le r$.*

If $u \in \tilde{\mathcal{O}}$ is nonzero then $u = t_1^{k_1} \cdots t_r^{k_r} v$ where $k_i = \nu_{p_i}(u)$ and v is invertible.

The lemma is proved in Shafarevich.

Lemma 2. *$\tilde{\mathcal{O}}$ is a f.g. $\mathcal{O}_{Y,q}$–module (comes from the fact that f is a finite map).*

Lemma 3. *$\tilde{\mathcal{O}}$ is a free $\mathcal{O}_{Y,q}$–module of rank $n = \deg(f)$. [$f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$–module of rank $n = \deg(f)$.]*

Proof of the theorem from Lemmas 1,2,3: Let $t =$ local parameter at q on Y. That is, $t \in m_{Y,q} \setminus m_{Y,q}^2$. So $t \in \tilde{\mathcal{O}}$ (really we mean f^*t but we ignore the f^* in the notation). So $t = t_1^{k_1} \cdots t_r^{k_r} v$ where $k_i = \nu_{p_i}(t)$ and v is invertible in $\tilde{\mathcal{O}}$. So $f^*(q) = \sum k_i p_i$. So $\deg f^*(q) = \sum_{i=1}^r k_i$.

$$\frac{\tilde{\mathcal{O}}}{(t)} = \frac{\tilde{\mathcal{O}}}{(t_1^{k_1})} \oplus \frac{\tilde{\mathcal{O}}}{(t_2^{k_2})} \oplus \cdots \oplus \frac{\tilde{\mathcal{O}}}{(t_r^{k_r})}.$$

It is easy to see that $\dim_k(\tilde{\mathcal{O}}/(t_i^{k_i})) = k_i$ (c.f $k[t]/(t^{k_i})$). So $\dim_k(\tilde{\mathcal{O}}/(t)) = \sum k_i = \deg f^*(q)$. Note $\mathcal{O}_{Y,q}/(t) = k$. But $\tilde{\mathcal{O}} = \mathcal{O}_{Y,q}^{\oplus \deg f}$. So

$$\frac{\tilde{\mathcal{O}}}{(t)} = \left(\frac{\mathcal{O}_{Y,q}}{(t)}\right)^n$$

of dimension n. So $n = \deg f^*(q)$□

Proof of Lemma 3 from Lemmas 1 and 2: $\tilde{\mathcal{O}}$ is a f.g. $\mathcal{O}_{Y,q}$–module. So $\tilde{\mathcal{O}} =$ (torsion)$\oplus$(free). But $\tilde{\mathcal{O}} \supset \mathcal{O}_{Y,q}$ sits inside $k(X)$, a field. So torsion part is zero. So

$\tilde{\mathcal{O}} = \mathcal{O}_{Y,q}^m$. We need $m = \deg(f)$.

$$\begin{array}{ccc} k(Y) & \subset & k(X) \\ \cup & & \cup \\ \mathcal{O}_{Y,q} & \subset & \tilde{\mathcal{O}} \end{array}$$

we want to show that $m = n = \deg(f)$. $n = [k(X) : k(Y)]$. $m =$ maximal number of elements of $\tilde{\mathcal{O}}$ linearly independent over $\mathcal{O}_{Y,q}$. So $m =$ maximal number of elements of $\tilde{\mathcal{O}}$ linearly independent over $k(Y) = \mathrm{Frac}(\mathcal{O}_{Y,q})$. So $m \leq n$. Let $\alpha_1, \ldots, \alpha_n \in k(X)$ be a basis of $k(X)$ over $k(Y)$. Let $t =$ parameter of Y at q (generates $m_{Y,q}$). Then $t^\ell \alpha_1, \ldots, t^\ell \alpha_n$ belong to $\tilde{\mathcal{O}}$ for ℓ sufficiently large. They are linearly independent over $\mathcal{O}_{Y,q}$, so $m \geq n$. Therefore $m = n\square$

11.1. **Bézout's Theorem.**

Theorem. *Let $X, Y \subset \mathbb{P}^2$ be projective curves. Suppose X is smooth, $X \nsubseteq Y$ (Y may be singular and may have several components).*

Then the sum of the multiplicities of the intersection of X and Y at all points of their intersection equals $\deg(X) \cdot \deg(Y)$.

Theorem (Bézout version 2)**.** *Let $X \subset \mathbb{P}^n$ be a smooth projective curve and $Y = V(F) \subset \mathbb{P}^n$ be a hypersurface such that $X \nsubseteq V(F) = Y$. Then the sum of the multiplicities of intersection of X and Y at all points of $X \cap Y$ equals* $(\deg(X))(\deg(Y))$.

To understand this, need the definitions of:

(1) sum of multiplicities of X and Y, denoted $X \cdot Y$ or $X \cdot F$.
(2) need $\deg(X)$.
(3) $\deg(Y) := \deg F$.

Definition. *$X \cdot F$ (or $X \cdot Y$) is* $\deg(\mathrm{div}_X F)$ *where* $\mathrm{div}_X F = i^* Y$, $i : X \hookrightarrow \mathbb{P}^n$.

Note $X \cdot F \geq 0$.

Definition. $\deg(X) := \max\{\#X \cap H : H \subset \mathbb{P}^n$ *is a hyperplane not containing* $X\}$ *(or, H is a hyperplane not containing any component of X, if X is reducible.)*

Homework problems on divisors on curves: pg. 174, nos. 2,3,4,6.

12. Bézout's theorem

$X \subset \mathbb{P}^n$ smooth curve. $F \in k[x_0, \ldots, x_n]$ homogeneous form of degree d such that $F \notin I_X$. We defined $X \cdot F$ to be the degree of the divisor of F on X. ($X \hookrightarrow \mathbb{P}^n$, $\mathrm{div}_X F = i^* \mathrm{div} F$ is the definition). Also, $\deg X = \max\{|X \cap H| : H$ is a hyperplane and $H \not\supseteq$ any component of $X\}$.

Remarks. (1) If $\deg F = \deg G$, $F, G \in k[x_0, \ldots, x_n]$ then $X \cdot F = X \cdot G$ since $\mathrm{div}_X(F/G)$ (henceforth denoted $\mathrm{div}(F/G)$) $= \mathrm{div} F - \mathrm{div} G$ has degree 0.

(2) Let $L =$ linear form not in I_X. Then $X \cdot F = X \cdot L^d = d(X \cdot L)$.

(3) $X \cdot L = \sum_{p \in X \cap V(L)} \nu_p(\mathrm{div} L)$.

Let $p \in X$ and L a linear form not in I_X. When is $\nu_p(\mathrm{div} L) = 0, 1, 2 \ldots$? $\nu_p(\mathrm{div} L) \geq 0$. When is $\nu_p(\mathrm{div} L) = 0$?

$\nu_p(\mathrm{div} L) = 0 \iff p \notin X \cap V(L) \iff L(p) \neq 0$.

$\nu_p(\mathrm{div} L) \geq 2$ iff $L(p) = 0$ and $T_{X,p} \subset T_{V(L),p} \subset T_{\mathbb{P}^n,p}$. This is a good exercise (and is in the book). Requires to think about tangent spaces. Add to homework list.

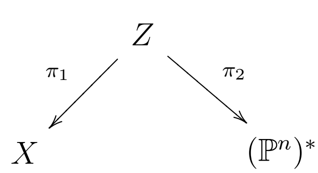
Theorem (Bézout's theorem). *Let $X \subset \mathbb{P}^n$ be a smooth projective curve, $F \in k[x_0, \ldots, x_n]$ homogeneous, $F \notin I_X$. Then $X \cdot F = (\deg X)(\deg F)$.*

Proof. Need only consider the case $F = L$ is linear. Show $X \cdot L = \deg X$. If we can find a linear form L such that $\nu_p(\mathrm{div} L) = 1$ for all $p \in X \cap V(L)$ then we are done, because $X \cdot L = \# X \cap L$. Suppose M is another linear form which gives the maximum value of $\# X \cap V(M) = \deg X$. Then $X \cdot L = X \cdot M \geq \# X \cap V(M)$. So $\# X \cap V(L) \geq \# X \cap V(M)$. So by maximality, $\# X \cap L = \# X \cap M = \deg X$.

To find such an L, consider $Z \subset X \times (\mathbb{P}^n)^*$ where $(\mathbb{P}^n)^*$ denotes the set of hyperplanes in $\mathbb{P}^n$. Let

$$Z = \{(p, V(L)) : V(L) \text{ is tangent to } X \text{ at } p\}.$$

Make sure you understand that this is algebraic. Exercise: find the equations for it.



$\pi_1^{-1}(p) = p \times \{V(L) : V(L) \supset T_{X,p}\}$. Since $T_{X,p}$ is a line, this is a linear space, so is irreducible and has dimension $n-2$.

X has dimension 1, fibres are all irreducible and have dimension $n-2$. So Z is irreducible and $\dim Z = (n-2)+1 = n-1$. Then $\pi_2(Z) =$ bad set of lines, and has dimension $\leq n-1$. So $\pi_2(Z) \neq (\mathbb{P}^n)^*$ and so we can find L such that $\nu_p(\mathrm{div} L) = 1$ for all $p \in X \cap V(L)$. □

12.1. **Elliptic curves.** Review: Chapter 1, Section 1.6, In particular, a smooth plane cubic $X \subset \mathbb{P}^2$ is not rational.

$X \subset \mathbb{P}^2$ smooth plane cubic.

Theorem. *Let $p_0 \in X$. The map*

$$X \to \mathrm{Cl}^0(X)$$

$$p \mapsto [p - p_0] := c_p$$

is a $1-1$ correspondence.

Proof. $1-1$: if $c_p = c_q$ then $p - p_0 \sim q - q_0$. If $p \neq q$ then $p \sim q \implies X$ is rational, which is a contradiction (so injectivity holds for any non-rational curve). To show onto, key point is given $p, q \in X$, show $\exists r \in X$ such that $p + q \sim r + p_0$, ie. $(p - p_0) + (q - p_0) \sim r - p_0$. Continued in next lecture. □

13. Elliptic curves

$X \subset \mathbb{P}^2$ smooth plane cubic. Let $p_0 \in X$.

$$X \to \mathrm{Cl}^0(X)$$

$$p \mapsto [p - p_0] = c_p$$

is a $1-1$ correspondence ($1-1$ was done last time). Key point for surjectivity is, given $p, q \in X$, $\exists r \in X$ such that $p + q \sim r + p_0$.

Proof of key point:

Case 1: Suppose $p \neq q$. Let L be the line through p and q. $\mathrm{div} L = p + q + s$ for some s by Bézout's theorem.

Case 2: $p = q$. Let $L =$ tgt line to X at p. Then $\mathrm{div} L = 2p + s$ (s could equal p).

Case a: $s \neq p_0$. Let M = line through s and p_0. Then div$M = s + p_0 + r$. Know $p + q + s \sim s + p_0 + r$ since div$L \sim$ div$M$.

Case b: $s = p_0$. Take M = tgt line through s. Then div$M = 2p_0 + r$ for some r, and $2p + p_0 \sim 2p_0 + r$. Now result follows easily.

The next claim is that if D is effective then there exists $p \in X$ such that $D \sim p + kp_0$.

Proof: Induction on deg$D$.

If deg$D = 1$ then $D = p$, so clear. If deg$D \geq 2$ then $D = D' + q$, D' effective. By induction, $D' \sim p = Kp_0$ for some p. So $D \sim p + q + kp_0$. But $p + q \sim r + p_0$ so $D \sim r + (k+1)p_0$. This proves the claim.

In general, if deg$D = 0$ then $D = D_1 - D_2$, D_i effective, deg$D_1 =$ degD_2. By the claim, $D_1 \sim p_1 + kp_0$, $D_2 \sim p_2 + kp_0$. So $D_1 - D_2 \sim p_1 - p_2$. Need to find a pt. $r \in X$ such that $p_1 - p_2 \sim r - p_0$, ie. $p_1 + p_0 \sim r + p_2$. Apply the "key point" from above, with p_0 replaced by p_2, so there does exist such an r. So we have shown that $X \to \mathrm{Cl}^0(X)$ is surjective □

Theorem. *Let $X \subset \mathbb{P}^2$ be a smooth plane cubic. Then*

$$\ell(D) = \deg(D) \qquad (*)$$

for all effective $D \in \mathrm{Div}(X)$, $D \neq 0$. Conversely, if $(*)$ holds for a smooth projective curve X, then X is isomorphic to a smooth plane cubic.

Proof. Fix $p_0 \in X$. We know $D \sim p + (d-1)p_0$, $d =$ deg$D$ for some $p \in X$. Consider

$$L(0) \subset L(p) \subset L(p + p_0) \subset \cdots \subset L(p + kp_0).$$

We have $L(0) = L(p) = k$ because X is not rational. The condition $(*)$ is equivalent to the inclusions after the first step being strict. Claim: $L(p + (\alpha-1)p_0) \subsetneq L(p + \alpha p_0)$ for all $\alpha \geq 1$.

$L(p) \subsetneq L(p + p_0)$. Let L_1 = line through p and p_0; div$L_1 = p + p_0 + r$. Let L_2 be a line through r that misses p and p_0 (or if $r = p$ or p_0, need something else). Then $\psi = L_2/L_1$ belongs to $L(p + p_0)$ (note that we also need to consider the case $p = p_0$ and the case $p \neq p_0$ but $r = p$ or $r = p_0$). *(Note also that here, since we are in $\mathbb{P}^2$, we are identifying a line with its defining equation, ie. writing the line L as $L = 0$)*

Claim: $\exists f_\ell \in L(\ell p_0)$ ($\ell \geq 2$) such that $\mathrm{div}_\infty f_\ell = \ell p + 0$.

If so, we are done since $f_\ell \in L(p + \ell p_0)$ and $f_\ell \notin L(p + (\ell - 1)p_0)$.

Need to find $x \in L(2p_0)$ and $y \in L(3p_0)$. If so, we are done, since $x^d \in L(2dp_0)$ and $x^{d-1}y \in L((2d+1)p_0)$ with poles of the correct order at p_0.

For $L(2p_0)$, let M_1 = tangent line to X at p_0. Then $\text{div}M_1 = 2p_0 + s$. Choose a line M_2 which goes through s with order 1. Then M_2/M_1 cancels the pole $1/M_1$ at s. $\text{div}M_2 = s + p + q$, $p, q \neq s, p_0$.

For $L(3p_0)$, let M_3 = line through p and p_0; $\text{div}M_3 = p + p_0 + t$ for some t. Then take M_4 = line of multiplicity one through t missing p_0, then

$$\frac{M_2}{M_1} \cdot \frac{M_4}{M_3}$$

has the right order of pole. □

14. The group law

If $(*)$ $\text{deg}D = \ell(D)$ for all $D > 0$ on a smooth projective curve X then X is isomorphic to a smooth plane cubic.

Proof. Suppose $(*)$ holds. Fix $p \in X$.

$$L(0) \subset L(p) \subset L(2p) \subset L(3p) \subset \cdots$$

Let $x \in L(2p) \setminus L(p)$. Let $y \in L(3p) \setminus L(2p)$. Define $\psi : X \dashrightarrow \mathbb{P}^2$ via $q \mapsto (1 : x(q) : y(q))$. This is a rational mapping. It is defined everywhere since X is a curve, but in this representation it is not defined at $q = p$ (in fact $p \mapsto (0 : 0 : 1)$). So $\psi : X \to \mathbb{P}^2$ regular.

$$L(p) 1$$

$$L(2p) 1, x$$

$$L(3p) 1, x, y$$

$$L(4p) 1, x, y, x^2$$

$$L(5p) 1, x, y, x^2, xy$$

$$L(6p) 1, x, y, x^2, xy, y^2, x^3$$

Since $\dim L(6p) = 6$ but we have seven elements, there is a linear dependence. So $\exists f(x, y) = 0$ on X and $\text{deg}f \leq 3$.

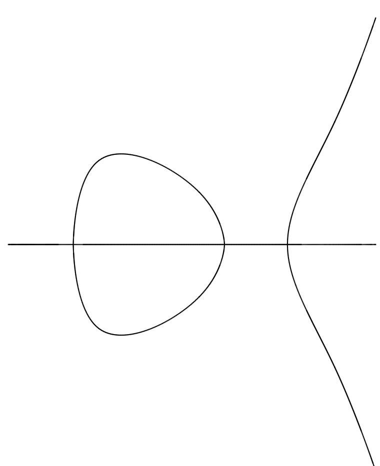
$\psi : X \to C \subset \mathbb{P}^2$, where $C =$ homogenisation of zero set of f. This is surjective because it is not constant. The curve C is either (a) a smooth cubic, (b) a singular cubic or (c) a conic. A singular cubic is birational to $\mathbb{P}^1$ (by passing lines through the singular point and projecting down) and so is a conic. Suppose we can prove $\deg\psi = 1$. Then X and C are birational. Then in cases (b) and (c), $X \cong \mathbb{P}^1$ because it is birational to $\mathbb{P}^1$. But $\deg(D) = \ell(D) - 1$ for a rational curve, so this is impossible. So we are in case (a), and $X \cong$ smooth plane cubic. We must therefore show that $\deg\psi = 1$.

$k(x), k(y) \subset k(C) \subset k(X)$. Since X is a curve, x and y are regular maps $X \to \mathbb{P}^1$. Now, $[k(X) : k(x)] = \deg x$ as a map, which equals $\deg(\mathrm{div}_\infty x)$ (since X and $\mathbb{P}^1$ are both smooth) which is 2. And $[k(X) : k(y)] = \deg(\mathrm{div}_\infty y) = 3$. This forces $[k(X) : k(C)]$ to divide 2 and 3, so $[k(X) : k(C)] = 1 = \deg\psi$. So $X \cong C$. □

14.1. **The group law.** Let $X =$ smooth plane cubic. Let $o \in X$ be an inflexion point. Assume $\mathrm{char} k \neq 2, 3$. Get Weierstrass form

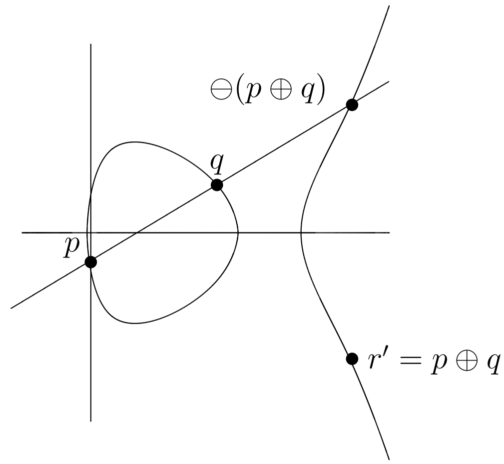
$$y^2 = x^3 + ax + b$$

(Note to self: not sure how to prove this using the fact that o is an inflexion point. Weierstrass form also obtainable by messy algebra.) Real part of X looks like:



$\mathrm{div}(z) = 3o$. If L is a line then if $\mathrm{div}L = p+q+r$, then in the group law on X, $(p-o)+(q-o) = p \oplus q - o$ for $p \oplus q \in X$. Then $p \oplus q \oplus r = o$ (three collinear points sum to zero). Denote the inverse in the group law by $\ominus p$; $p \oplus (\ominus p) = o \iff (p - o) + (\ominus p - o) \sim o$, so $p + (\ominus p) \sim 2o$. Geometrically if $q = \ominus p$ then line $\overline{po} \cap X = q + p + o$.

In $y^3 = x^3 + ax + b$ case, what is $\ominus(x, y)$?



Answer: $\ominus(x, y) = (x, -y)$.

Vertical lines = lines passing through o = point at ∞.

Inverse map $i : X \to X$, $p \mapsto \ominus p$ is a regular mapping.

Multiplication map $\psi : X \times X \to X$, $(p, q) \mapsto p \oplus q$. In coordinates, $(x_1, y_1) \times (x_2, y_2) \mapsto (x_3, y_3)$. Line through (x_1, y_1) and (x_2, y_2) has $m = \frac{y_2 - y_1}{x_2 - x_1}$, $y - y_1 = m(x - x_1)$. Subs. $y = m(x - x_1) + y_1$ in cubic gives cubic equation in x. But x_1, x_2 are known to be solutions. Then the third solution can be computed from the x^2 term. $x_3 = m^2 - x_1 - x_2$ is the third root, and $y_3 = m(x_3 - x_1) + y_1$. This is a rational map.

Theorem. *$\psi : X \times X \to X$ is a regular map.*

Example. Translation. For each $q \in X$, define $t_q : X \to X$ by $p \mapsto p \oplus q$. This is a regular map because addition is. It is also an iso. because the inverse is $t_{\ominus q}$.

Proposition. *$d\psi_{(p,q)} : T_{(p,q), X \times X} \to T_{p \oplus q, X}$ equals $dt_q \oplus dt_p$, where $T_{(p,q), X \times X}$ is identified with $T_{p,X} \oplus T_{q,X}$. In particular, $d\psi_{(p,q)}$ is surjective.*

15. Solutions and partial solutions to selected exercises

(1) Page 174, problem 3. Prove that the number of singular points of an irreducible plane curve of degree n is $\leq \binom{n-1}{2}$.

Let C be a plane curve of degree n. Suppose C has $\binom{n-1}{2} + 1$ singular points. It is a fact that $\binom{n+2}{2} - 1$ points uniquely determine a plane curve of degree n (by considering the span of the monomials of degree n). Choose a plane curve X that intersects C in $\binom{n-1}{2}$ singular points and $\binom{n+2}{2}-1-\left(\binom{n-1}{2}+1\right)=3n-2$ nonsingular points (which we are free to choose). We can find an X that does this. We can use Bézout to get $C\cdot X=n^2$. Each singular point gives intersection with multiplicity ≥ 2, so $C\cdot X\geq 2\left(\binom{n-1}{2}+1\right)+(3n-2)=n^2+2$.

Here we have also used Problem 2 (the normalisation) to get that $C\cdot X$ is well-defined for a singular C, and that each singular point has multiplicity ≥ 2.

(2) Page 174, problem 4. Hessian

$$H=\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix}$$

$\deg H=3n-2$ if $\deg X=n$, so $H\cdot X=3n(n-2)$. Still need to know that the order of vanishing of H at $x=$ (multiplicity of tgt line at x) -2. To prove this, try using the Taylor series (and take $x=(0,0)$).

(3) **Pencil of conics.** $X\xrightarrow{\pi}\mathbb{P}^1$. Idea: over each point of $\mathbb{P}^1$ is a conic, such that:

- $\pi^{-1}(\infty)$ is nonsingular.
- Over $\mathbb{A}^1_t=\mathbb{P}^1\setminus\{\infty\}$, each fibre is the variety determined by $\sum_{i,j=0}^{2}a_{ij}(t)x_ix_j$.
- X is nonsingular.

eg. $tx^2+(y^2-z^2)$ is a nonexample - $\pi^{-1}(\infty)$ is a double line, so singular. An example is the subset of $\mathbb{P}^1_{st}\times\mathbb{P}^2_{xyz}$ determined by $t(x^2+y^2+z^2)+sxy$. Each zero of

$$\det\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

gives t such that $\pi^{-1}(t)$ is singular (in the scheme-theoretic sense), ie. a singular conic or a double line, since the rank of the coefficient matrix determines whether the conic is singular or not.

rank 3	$x^2+y^2+z^2$	
rank 2	x^2+y^2	line pair
rank 1	x^2	double line.

The group $\mathrm{Cl}(X)$ is generated by the following. First, $[F]$ for F some nonsingular fibre. Next, each singular fibre splits into $L_1 \cup L_2$; pick one of them to get the set $\{[L_1] : L_1 + L_2 \text{ is a singular fibre}\}$. Tsen's theorem implies that there is a section of π (see Shafarevich for proof). Let $\sigma : \mathbb{P}^1 \to X$ be a section and take the class $[S]$ where $S = \sigma(\mathbb{P}^1)$.

Since $X \to \mathbb{P}^1$, any two fibres are linearly equivalent (fibre $= \pi^{-1}$(pt.).

Claim 2: the given generators are not linearly equivalent.

Suppose I have $D = m[F] + n[S] + \sum \ell_i[L_i] = 0 \in \mathrm{Cl}(X)$ where the sum is over the different singular fibres. Choose $F' \neq F$ nonsingular. Look at $i^*(D)$ on F'. Since $F \cap F' = \varnothing$, $i^*[F] = 0$. Similarly, $i^*[L_i] = 0$. Then $i^*n[S] = np$, $p \in F'$, $F' \cap S = \{p\}$, so $n = 0$.

Then look at $j :$ other line $L_i' \hookrightarrow X$. Look at $j^*(D)$ and get $\ell_i = 0$ for all i. Use that $m[F] = 0$ implies $m = 0$ because X is projective; if $m \neq 0$ then we can assume $m > 0$ and there can't be any nonzero functions which are regular and vanish along F, since any regular function is a constant.

We still need to show that any prime divisor is linearly equivalent to a sum of the given ones. See Shafarevich, pp. 73 and 164 for this example worked out in detail.

16. Theorem of the square

Recall some facts about ramification and the Frobenius map.

(1) If $f : X \to Y$ regular, $\mathrm{char} k = 0$, X smooth.
   (a) Fix $q \in Y$. If $df_p : T_{p,X} \to T_{q,Y}$ is surjective for all $p \in f^{-1}(q)$ then $f^{-1}(q)$ is smooth.
   (b) If Y is a smooth curve and df_p is surjective for all $p \in f^{-1}(q)$ then $f^*(q) = \sum(\text{components of } f^{-1}q)$, ie. each component appears with multiplicity one.

(2) Suppose $f : X \to Y$ is finite, X, Y irred, Y normal ($\Longleftarrow$ smooth).
   (a) **Theorem.** $\#f^{-1}(q) \leq \deg(f)\ \forall q \in Y$.
   (b) f is called *unramified over* q if $\#f^{-1}(q) = \deg f$.
   (c) If $k(X) \hookrightarrow k(Y)$ is separable then $\{q \in Y : f \text{ is unramified over } q\}$ is a nonempty open set in Y (& therefore dense).

Note: if $f : X \to Y$ is a morphism of smooth curves then $f^*(q) = \sum_{p \in f^{-1}(q)} p$ for unramified q.

(3) (Frobenius)

Let X = affine variety defined over $\mathbb{F}_p$ (ie. equations of generators of I_X have coefficients in $\mathbb{F}_p$, but X is really in $\mathbb{A}^n = (\overline{\mathbb{F}_p})^n$.

Frobenius

$$\varphi : X \to X$$

$$(a_1, a_2, \ldots, a_n) \mapsto (a_1^p, a_2^p, \ldots, a_n^p)$$

(since if $f(a_1, \ldots, a_n) = 0$ then $f(a_1^p, \ldots, a_n^p) = 0$, $a_i \in \overline{\mathbb{F}_p}$).

$\varphi : X \to X$ is a regular finite map.

This holds also if X is projective or quasiprojective.

Two facts:

(1) If X is a curve then $\deg\varphi = p$.

(2) If $f : X \to Y$ is a map of curves, not separable, and X, Y defined over $\mathbb{F}_p$, then $\exists$ a regular map $g : X \to X$ such that $f = g \circ \varphi$.

16.1. **Theorem of the square.** X = smooth plane cubic. $o \in X$ as before (pt. at ∞ and inflexion point).

$$\Delta = \{(p,p) : p \in X\} \subset X \times X$$

$$\Sigma = \{(p, \ominus p) : p \in X\}$$

Theorem. *On $X \times X$,*

$$\Delta + \Sigma \sim 2(o \times X + X \times o)$$

Proof. Idea is to find a $g \in k(X \times X)^*$ such that $\text{div}_0(g) = \Sigma + \Delta$ and $\text{div}_\infty(g) = 2(o \times X + X \times o)$.

$X \subset \mathbb{P}^2$. On affine part $U \subset X$, $y^2 = x^3 + ax + b$.

$U \times U \subset X \times X$, $U \times U \subset \mathbb{A}^4$. Coordinates $(x_1, x_2), (y_1, y_2)$ with

$$y_1^2 = x_1^3 + ax_1 + b \qquad (1)$$

$$y_2^2 = x_2^3 + ax_2 + b \qquad (2)$$

Let $g = x_1 - x_2 \in k(X \times X)^*$. Then g is regular on $U \times U$ so can only have poles on $X \times X \setminus U \times U = X \times o \cup o \times X$. In fact, $\mathrm{div}_\infty(g) = 2(o \times X + X \times o)$. Set-theoretically, $\div_0(g) = a\Delta + b\Sigma$. Show coefficients are 1. From (1) and (2), get $(y_1 - y_2)(y_1 + y_2) = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + a)$. Then $x_1 - x_2$ is local equation of both Δ and Σ. For example, local equation of Δ in open set where $y_1 + y_2 \neq 0$, $y_1 - y_2 = (x_1 - x_2)\frac{(\ldots)}{(y_1+y_2)}$, so local equation of Δ is $x_1 - x_2$. In this way, we get that coefficients of Δ and Σ are 1. □

16.2. **Regular maps $X \to X$.** If $q \in X$, $t_q : X \to X$ defined by $p \mapsto p \oplus q$. If $\lambda' : X \to X$ is any regular map, then if $\lambda'(0) = q$ then $t_{\ominus q} \circ \lambda'$ is also a regular map and maps $o \mapsto o$. Form a group G.

$$G = \{\lambda : X \to X : \lambda \text{ is regular and } \lambda(o) = o\}$$

with $(\lambda_1 + \lambda_2)(x) = \lambda_1(x) \oplus \lambda_2(x)$ and $(-\lambda)(x) = \ominus\lambda(x)$. This is a well-defined group structure on G because $\oplus : X \times X \to X$ and $\ominus : X \to X$ are regular maps. G is an abelian group with identity the constant map.

Definition. *For $\lambda \in G$, let*

$$n(\lambda) = \begin{cases} 0 & \lambda \text{ constant map} \\ \deg\lambda & \text{otherwise} \end{cases}$$

Theorem. *For all $\lambda, \mu \in G$,*

$$n(\lambda + \mu) + n(\lambda - \mu) = 2(n(\lambda) + n(\mu)).$$

17. Hasse-Weil estimates

X = smooth plane projective cubic curve; o = inflexion point and point at ∞. $G = \{\lambda : X \to X : \lambda \text{ regular map}, \lambda(o) = o\}$. G is an abelian group under pointwise addition. For $\lambda \in G$,

$$n(\lambda) = \begin{cases} 0 & \lambda = 0 \\ \deg\lambda & \text{otherwise} \end{cases}$$

17.1. **Scalar products on a group G.** If $\lambda, \mu \in G$, $(\lambda, \mu) \in \mathbb{Q}$ such that $(\lambda, \mu) = (\mu, \lambda)$ and $(\lambda_1 + \lambda_2, \mu) = (\lambda_1, \mu) + (\lambda_2, \mu)$. How to specify such a scalar product? Suppose we have a $\mathbb{Q}$–valued function $n(\lambda) \in \mathbb{Q}$, $n(\lambda) \geq 0$, $n(o) = 0$. We want a scalar product such the $n(\lambda) = (\lambda, \lambda)$. Need:

$$n(\lambda + \mu) = n(\mu) + 2(\mu, \lambda) + n(\lambda)$$

$$n(\mu - \lambda) = n(\mu) - 2(\mu, \lambda) + n(\lambda)$$

so $n(\mu + \lambda) + n(\mu - \lambda) = 2(n(\mu) + n(\lambda))$. Simple fact: if $n(\cdot)$ satisfies this equation, then $\frac{1}{2}(n(\lambda + \mu) - n(\lambda) - n(\mu))$ is a scalar product on G.

Theorem. *For all $\lambda, \mu \in G$,*

$$n(\lambda + \mu) + n(\lambda - \mu) = 2(n(\lambda) + n(\mu)).$$

Corollary. *There exists a scalar product on G such that $(\lambda, \lambda) = n(\lambda)$.*

Proof. Fix λ, μ. Define $f : X \to X \times X$ via $p \mapsto (\lambda(p), \mu(p))$. We have four maps $X \times X \to X$:

$$\psi : (p, q) \mapsto p \oplus q$$

$$\varphi : (p, q) \mapsto p \ominus q$$

$$\pi_1 : (p, q) \mapsto p$$

$$\pi_2 : (p, q) \mapsto q$$

Theorem of the square says on $X \times X$, $\Delta + \Sigma \sim 2(o \times X + X \times o) = 2(\pi_1^*(o) + \pi_2^*(o))$. Apply f^*:

$$f^*\Delta + f^*\Sigma \sim 2(f^*\pi_1^*(o) + f^*\pi_2^*(o)) \qquad (*)$$

$f^*\pi_1^*(o) = (\pi_1 f)^*(o) = \lambda^*(o)$ if $\lambda \neq 0$.

$f^*\pi_2^*(o) = (\pi_2 f)^*(o) = \mu^*(o)$ if $\mu \neq 0$.

(We don't need to worry about the case λ or $\mu = 0$ because then it becomes $n(\mu)+n(-\mu) = 2n(\mu)$ so $n(\mu) = n(-\mu)$. But $\ominus$ is an automorphism $\implies$ degree 1, and so this is true.)

$f^*\Delta = f^*\varphi^*(o) = (\varphi f)^*(o) = (\lambda - \mu)^*(o)$ if $\lambda - \mu \neq 0$.

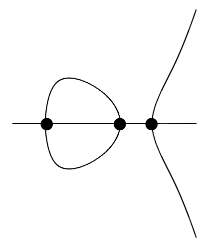
$f^*\Sigma = f^*\psi^*(o) = (\psi f)^*(o) = (\lambda + \mu)^*(o)$ if $\lambda + \mu \neq 0$.

So suppose $\lambda - \mu, \lambda + \mu \neq 0$. Apply $\deg(\cdot)$ to $(*)$. This gives the result.

Other cases: assume eg. $\lambda + \mu = 0$, ie. $\lambda = -\mu$. Need to prove $n(2\lambda) = 2(n(\lambda) + n(-\lambda))$. So $n(2\lambda) = 4n(\lambda)$.

$$2\lambda : X \xrightarrow{\lambda} X \xrightarrow{\cdot 2} X$$

$n(2\lambda) = \deg(2\lambda)^*(o) = n(\lambda)n(2\cdot)$. Need to compute $n(2\cdot) = \deg(2\cdot)^*(o)$. But $(2\cdot)^*(o) = o \cup (X \cap \{y = 0\})$ since $2x = 0$ iff $x = \ominus x$ iff y–coord of $x = -(y$–coord of $x)$.



Points x with 2x=0

□

17.2. **Hasse-Weil estimate.** $X \subset \mathbb{P}^2$ smooth plane cubic such that equation of X has coefficients in $\mathbb{F}_p$.

Let $\varphi : X \to X$ send (x, y, z) to (x^p, y^p, z^p), the Frobenius. We know that $n(\varphi) = p$. Let $X(\mathbb{F}_p) = \{(x, y, z) \in X : x, y, z \in \mathbb{F}_p\}$. Let $N = \#X(\mathbb{F}_p)$. What is $n(1 - \varphi)$?

$$1 - \varphi : X \to X$$
$$\alpha \mapsto \alpha \ominus \varphi(\alpha).$$

Note $\alpha = \varphi(\alpha)$ if and only if $\alpha \in X(\mathbb{F}_p)$. So $(1 - \varphi)^{-1}(o) = X(\mathbb{F}_p)$.

Claim.

$$(1 - \varphi)^*(o) = \sum_{\alpha \in X(\mathbb{F}_p)} \alpha.$$

Consider $(1-\varphi)^{-1}(\alpha)$ and $(1-\varphi)^{-1}(o) = \{\gamma_1, \ldots, \gamma_N\}$. If $\beta \ominus \varphi(\beta) = \alpha$ then $(1-\varphi)^{-1}(\alpha) = \{\beta+\gamma_1, \ldots, \beta+\gamma_N\}$. If $1-\varphi$ is separable then $1-\varphi$ is unramified over some α, and therefore $\deg(1 - \varphi) = N$. Recall that $1 - \varphi$ not separable $\implies 1 - \varphi = \mu \circ \varphi \implies 1 = \varphi + \mu\varphi$, $\mu : X \to X$ some regular map. So $1 = (1 + \mu) \circ \varphi$ where "$+$" = sum in group law. So

$$1 = \deg 1 = \deg(1 + \mu)\deg\varphi = \deg(1 + \mu)p,$$

a contradiction. So $1 - \varphi$ is separable and we obtain the claim.

The theorem says:

$$n(1+\varphi)+n(1-\varphi)=2(n(1)+n(\varphi))$$
$$n(1+\varphi)+N=2(1+p)$$
$$n(1+\varphi)=2+2p-N.$$

Consider $\lambda : X \to X$ defined by

$$\lambda : \alpha \mapsto \underbrace{\alpha \oplus \cdots \oplus \alpha}_{a} \oplus \underbrace{\varphi(\alpha) \oplus \cdots \oplus \varphi(\alpha)}_{b}$$

ie. $\lambda = a + b\varphi$, $a, b \in \mathbb{Z}$, "+" in group law. Then

$$n(\lambda) = n(a+b\varphi) = (a+b\varphi, a+b\varphi) \geq 0 \qquad \forall a, b \in \mathbb{Z}.$$

Since $(a+b\varphi, a+b\varphi) = a^2 + 2ab(1,\varphi) + b^2$, this holds if and only if $\begin{pmatrix} 1 & (1,\varphi) \\ (1,\varphi) & 1 \end{pmatrix}$ is positive semidefinite $\iff p \geq (1,\varphi)^2$. So we get $|(1,\varphi)| \leq \sqrt{p}$.

18. Differential forms

X = smooth plane cubic in $\mathbb{P}^2$, $\text{char}k \neq 2, 3$ ($\text{char}k < \infty$).

(1) Have scalar product $(-,-) : G \times G \to \mathbb{Q}$ where $G = \{\lambda : X \to X$ regular such that $\lambda(o) = o\}$, such that $(\lambda, \lambda) = n(\lambda) = \deg(\lambda)$ for λ non-constant.

(2) Had: Frobenius map $\varphi : X \to X$

$\deg\varphi = p$.

$\deg(1-\varphi)^*(o) = N = \#X(\mathbb{F}_p)$.

(3) used: $\lambda = a + b\varphi$, $a, b \in \mathbb{Z}$, ie. $\lambda(p) = ap \oplus b\varphi(p) \in X$.

Get $(1,\varphi)^2 \leq p$, so $|(1,\varphi)| \leq \sqrt{p}$.

Put it all together:

$N = n(1-\varphi) = 1 - 2(1,\varphi) + p$. Therefore, $N - p - 1 = -2(1,\varphi)$. So

$$|N-p-1| = 2|(1,\varphi)| \leq 2\sqrt{p}.$$

Example. $\text{char}k = 7$.

$|N-8| \leq 2\sqrt{7} = 5.23\ldots$. So $|N-8| \leq 5$. So $N \in \{3, 4, \ldots, 13\}$.

$y^2 = x^3 + ax + b$, $a, b \in \mathbb{F}_7$.

49 curves possible, 42 smooth ones. Computer calculation:

N	# of curves
3	1
4	4
5	3
6	6
7	4
8	6
9	4
10	6
11	3
12	4
13	1

$y^2 = x^3 + 3$ has 13.

What about char$k = 0$? The Mordell-Weil theorem says that $X(\mathbb{Q})$ is a f.g. abelian group. It is not known whether $X(\mathbb{Q})$ has bounded rank. Current record is about 29.

18.1. **Differential forms.** Situation: let X be a variety and $p \in X$. Ring of regular functions at p is $\mathcal{O}_{X,p} \supset m_{X,p} =$ those which vanish at p. $T_{X,p} = (m_{x,p}/m_{X,p}^2)^*$ tangent space to X at p. Given $f \in \mathcal{O}_{X,p}$, define $d_p f \in T_{X,p}^* = m_{X,p}/m_{X,p}^2$, the linear part of the Taylor series of f at p. Or take $(f - f(p)) + m_{X,p}^2$; the same thing.

Want to do now: study how $d_p f$ varies with p.

Definition. *Let*

$$\Phi[X] = \{\text{functions } \phi : X \to \sqcup_{x \in X} T_{X,x}^* \text{ such that } \phi(p) \in T_{X,p}^* \text{ for all } p\}$$

(set of sections of $\sqcup_{p \in X} T_{X,p}^* \to X$*).*

Then $\Phi[X]$ is an abelian group under $(\phi+\psi)(p) = \phi(p)+\psi(p)$, and $\Phi[X]$ is a $k[X]$–module.

Definition. *Given* $f \in k[X]$*, define* $df \in \Phi[X]$ *via* $df(p) = d_p f \in T_{X,p}^*$.

Definition. *$\varphi \in \Phi[X]$ is called a regular differential form on X if for every $p \in X$, there exists an open neighbourhood U of p such that $\varphi|_U$ is in the $k[U]$–submodule of $\Phi[U]$ generated by the image of $k[U] \to \Phi[U]$, $f \mapsto df$.*

Unravelling this, on U,

$$\varphi|_U = \sum_{i=1}^{r} f_i dg_i, \qquad f_i, g_i \in k[U].$$

Definition. *Let $\Omega[X] = k[X]$–module consisting of regular differential forms on X.*

(Aside: if $U \subset X$ open, define $\Omega_X(U) = k[U]$–submodule consisting of all regular differential forms on U – Ω_X is a sheaf. If $V \subset U \subset X$ open,

$$\Omega_X(U) \to \Omega_X(V)$$

$$\varphi \mapsto \varphi|_V.)$$

18.2. **Rules involving df.** $f, g \in k[X]$.

$$d(f+g) = df + dg$$

$$d(fg) = fdg + gdf$$

Chain rule: if $F = F(T_1, \ldots, T_m) \in k[T_1, \ldots, T_m]$ and if $f_1, \ldots, f_m \in k[X]$ then $F(f_1, \ldots, f_m) \in k[X]$ and

$$dF(f_1, \ldots, f_m) = \sum_{i=1}^{m} \frac{\partial F}{\partial T_i}(f_1, \ldots, f_m) df_i.$$

First example: $X = \mathbb{A}^n$. $k[X] = k[x_1, \ldots, x_n]$. $dx_1, \ldots, dx_n \in \Omega[X]$. For each p, $d_p x_1, \ldots, d_p x_n \in m_p/m_p^2$ and form a basis of this vector space. So

$$\Phi[X] = \{\varphi = \sum_{i=1}^{n} \varphi_i dx_i \text{ s.t. the } \varphi_i \text{ are } \textit{any} \text{ functions } X \to k\}.$$

Suppose $\varphi \in \Omega[X]$. Let $p \in \mathbb{A}^n$ and $U \subset \mathbb{A}^n$ be the open set $U \ni p$ from the definition of φ. Then $\varphi|_U = \sum_{i=1}^{r} f_i dg_i$ where $f_i, g_i \in k[U]$. Then $dg_i = \sum \frac{\partial g_i}{\partial x_j} dx_j$. So get $\varphi|_U = \sum_{i=1}^{n} h_i dx_i$, h_i regular at p (combination of f_i and $\frac{\partial g_i}{\partial x_j}$). This implies $\varphi_i|_U = h_i$ (since the φ_i are uniquely determined by φ). So the φ_i are regular on all of X. So $\varphi_i \in k[X]$. So $\varphi = \sum_{i=1}^{n} \varphi_i dx_i$. Therefore,

$$\Omega[\mathbb{A}^n] = \bigoplus_{i=1}^{n} k[x_1, \ldots x_n] dx_i,$$

a free $k[x_1, \dots, x_n]$–module of rank n.

Next time:

- $\mathbb{P}^1$ (try it first on your own), $\Omega[\mathbb{P}^1]$.
- $x_0^3 + x_1^3 + x_2^3 = 0$. Find $\Omega[X]$.

19. Differential forms II

Homework problems:

p.188 # 2,4,5 plus 9 if interested.

pp. 204-205 # 6,7,9. and compute $\Omega[X]$, $X = V(x_0^3+x_1^3+x_2^3)$. (Can you do $x_0^n+x_1^n+x_2^n$?)

X variety, $\Phi[X] = \{\varphi : X \to \sqcup_{p\in X} T^*_{X,p} : \varphi(p) \in T^*_{X,p}\}$.

$$\Omega[X] = \{\varphi \in \Phi[X] \text{ s.t. locally (on } U\text{)}, \varphi|_U = \sum_{i=1}^{r} f_i dg_i, f_i, g_i \in k[U]\}.$$

Then $\varphi(p) = \sum f_i(p) d_p g_i \in T^*_{X,p}$. Showed $\Omega[\mathbb{A}^n] = \oplus_{i=1}^n k[\mathbb{A}^n]dx_i$.

Example. $X = \mathbb{P}^1_{xy}$. Two open sets $U_0 : x \neq 0$, $t = y/x$ affine coordinate on U_0. $U_1 : y \neq 0$, $s = x/y$ affine coordinate on U_1.

Suppose $\varphi \in \Omega[\mathbb{P}^1]$, $\alpha = \varphi|_{U_0} = p(t)dt$, p a polynomial in $k[t]$. $\beta = \varphi|_{U_1} = q(s)ds$, $q(s) \in k[s]$.

Need $\varphi|_{U_0\cap U_1}$ to be well-defined, ie. $\alpha|_{U_0\cap U_1} = \beta|_{U_0\cap U_1}$. So $p(t)dt = q(s)ds$, $k[U_0 \cap U_1] = k[s, \frac{1}{s}]$, $t = \frac{1}{s}$.

$ds = d(\frac{1}{t}) = -\frac{1}{t^2}dt$, so $p(t)dt = -\frac{1}{t^2}q(\frac{1}{t})dt$. This is true iff $p(t) = -\frac{1}{t^2}q(\frac{1}{t})$ on $U_0 \cap U_1$. This can't happen because if $q^*(t) := t^{\deg q}q(\frac{1}{t})$ then $t^{\deg q+2}p(t) = -q^*(t)$, $q^*(0) \neq 0$. But this is impossible. So $p(t) = q^*(t) = 0$ and $\Omega[\mathbb{P}^1] = 0$.

Example. Let $X \subset \mathbb{P}^2$, $x_0^3+x_1^3+x_2^3 = 0$, $\operatorname{char} k \neq 3$. We will find a nonzero element of $\Omega[X]$. Let $U_{ij} = X \setminus (V(x_i) \cup V(x_j))$. Then $X = U_{01} \cup U_{02} \cup U_{12}$. On U_{01}, let $x = x_1/x_0$, $y = x_2/x_0$ and consider $\varphi = dy/x^2$. On U_{12}, let $u = x_2/x_1$, $v = x_0/x_1$ and consider $\psi = dv/u^2$. On U_{02}, let $s = x_0/x_2$, $t = x_1/x_2$ and consider $\chi = dt/s^2$. Need to check that on U_{012}, these all agree.

$k[U_{01}] = k[x,y]/(x^3+y^3+1 = 0)$. Note that, in $\Omega[U_{01}]$, $d(x^3+y^3+1) = 0$, so $3x^2dx+3y^2dy = 0$. So $dx = -\frac{y^2}{x^2}dy$. On U_{12}, $v = 1/x$, $u = y/x$ and $dv/u^2 = (x^2/y^2)d(1/x) = \frac{x^2}{y^2} \cdot \frac{-1}{x^2}dx = \frac{-1}{y^2}dx = \frac{1}{x^2}dy$. Similarly, dt/s^2 is equal to the other two on the overlap. So $\Omega[X] \neq 0$.

Two simple remarks:

(1) Suppose $x_1, \ldots, x_n \in k[X]$ are such that $d_p x_1, \ldots, d_p x_n$ form a basis for $T^*_{X,p}$ for all $p \in X$. Then if $\varphi \in \Phi[X]$ or $\Omega[X]$, then $\exists$ unique functions $\alpha_1, \ldots, \alpha_n : X \to k$ such that $\varphi = \alpha_1 dx_1 + \cdots + \alpha_n dx_n$.

(2) If $\alpha : X \to k$ is a function such that $\exists$ open cover $\{U_i\}$ of X with $\alpha|_{U_i} \in k[U_i]$ for every i, then α is regular.

Theorem. *Let $p \in X$ be a smooth point. Then $\exists$ an open affine neighbourhood U of p such that $\Omega[U]$ is a free $k[U]$–module of rank $\dim_p X$.*

Proof. WLOG X irred (since only one component passes through a smooth point) and affine. $X \subset \mathbb{A}^N$, $I_X = (F_1, \ldots, F_m)$, $n = \dim_p X$.

$$\sum_{j=1}^{N} \frac{\partial F_i}{\partial x_j} dx_j = 0 \qquad \text{in } \Omega[X]$$

key: consider $\left(\frac{\partial F_i}{\partial x_j}(p)\right)$ has rank $N-n$. Rename the x's and F's such that $J = \det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,N-n \\ j=n+1,\ldots,N}}$ has $J(p) \neq 0$.

Let $U = X \setminus V(J)$. On U, $dx_1, \ldots, dx_n$ generate $T^*_{X,q}$ for all $q \in U$. Therefore, they are a basis. If $\varphi \in \Omega[U]$, $\varphi = \sum \alpha_i dx_i$. Show $\alpha_i \in k[U]$. We know $\exists$ an open cover $\{V_j\}$ of U such that $\varphi|_{V_j}$ has "nice form". So if $V = V_j$ then $\varphi|_V = \sum f_i dg_i$ where $f_i, g_i \in k[V]$. But $dg_i = \sum_{j=1}^{N} \frac{\partial g_i}{\partial x_j} dx_j = \sum_{j=1}^{n} h_j dx_j$, $h_j \in k[V]$ (check). This implies $\alpha_i \in k[V]$. $\square$

20. Kähler differentials and higher forms

20.1. **Differential forms.** X = smooth variety. $\Omega[X] \subset \Phi[X]$.

• Given $p \in X$ smooth, there exists open neighbourhood $X \supset U \ni p$ such that $\Omega[U]$ = free $k[U]$–module of rank $n = \dim_p X$. To show this, find $x_1, \ldots, x_n$ such that on U, $d_p x_1, \ldots, d_p x_n$ form a basis.

• algebraic definition (Kähler differentials).

• p-forms.

• rational forms.

20.2. **Kähler differentials.** Let X be affine, $A = k[X]$, $\Omega = \Omega[X]$, which is an f.g. A–module.

Definition. *Let $\Omega_{A/k}$ be the following A–module.*

$$\Omega_{A/k} = T/I$$

where $T =$ free A–module generated by df for all $f \in A$ and I is the submodule generated by

$$d(f+g) = df + dg$$
$$d(fg) = fdg + gdf$$
$$d(\alpha) = 0$$

for all $f, g \in A$ and all $\alpha \in k$.

Note that $\Omega_{A/k}$ is generated by $\{dx_1, \dots, dx_n\}$ if $A = k[x_1, \dots, x_n]/J$. Note there exists a homomorphism of A–modules $\Omega_{A/k} \xrightarrow{\alpha} \Omega$ defined by $df \mapsto (p \mapsto d_p f)$.

Proposition. *α is surjective.*

Proof. Let $\omega \in \Omega$. For $p \in X$, let $U_p =$ affine open nbd of p such that $\omega|_{U_p} = \sum_{i=1}^{r_p} f_{i,p} dg_{i,p}$, with $f_{i,p}, g_{i,p} \in k[U_p] \subset k(X)$. By clearing denominators, $\beta_p \omega|_{U_p} = \sum_{i=1}^{s=s(p)} r_{i,p} dh_{i,p}$ where $\beta_p, r_{i,p}, h_{i,p} \in A$. Choose points $p_1, \dots, p_\ell \in X$ such that $U_{p_1} \cup \cdots \cup U_{p_\ell} = X$. We know β_p does not vanish on U_p. Consider the ideal $(\beta_{p_1}, \dots, \beta_{p_\ell}) \subset A$. The Nullstellensatz implies $(\beta_{p_1}, \dots, \beta_{p_\ell}) = A$, so $\exists \gamma_i \in A$ such that $\sum_{i=1}^{\ell} \gamma_i \beta_{p_i} = 1$. Therefore,

$$\omega = \alpha \left(\sum_{j=1}^{\ell} \sum_{i=1}^{s(p_j)} \gamma_j r_{i,p_j} dh_{i,p_j} \right).$$

$\square$

Note that if $\omega \in \Omega[X]$ and $\omega(p) = 0$ for all $p \in U \subset X$ dense open, then $\omega = 0$.

Proposition. *If X is smooth and affine and irreducible then $\alpha : \Omega_{A/k} \to \Omega[X]$ is an isomorphism.*

Proof. A worthwhile exercise. See proof of previous prop. $\square$

20.3. **Differential r–forms.** Let

$$\Phi^r[X] = \{\varphi : X \to \sqcup_{p\in X} \wedge^r T^*_{X,p} : \varphi(p) \in \wedge^r T^*_{X,p} \forall p\}.$$

$\Phi^0[X]$ = ring of k–valued functions.

$\Phi^1[X] = \Phi[X]$.

Recall:

If V is a vector space with basis $e_1, \dots, e_n$ then $\wedge^r V$ has basis $e_{i_1} \wedge \cdots \wedge e_{i_r}$, $i_1 < \cdots < i_r$. Let $e_i \wedge e_j = -e_j \wedge e_i$. Can define $\varphi \wedge \psi$ for $\varphi \in \Phi^r[X]$, $\psi \in \Phi^s[X]$. Each $\Phi^r[X]$ is a $k[X]$–module.

Definition. $\Omega^r[X]$ = *set of regular r–forms on X. ω is a regular r–form if for all $p \in X$, $\exists$ open $U \ni p$ with $\omega|_U \in k[U]$–module generated by $df_1 \wedge \cdots \wedge df_r$ with $f_i \in k[U]$.*

Theorem. *If $p \in X$ is a smooth point and $n = \dim_p X$ then $\exists$ open neighbourhood U of p such that $\Omega^r[U]$ is a free $k[U]$–module of rank $\binom{n}{r}$.*

Proof. Identical to $r = 1$ case. $\Omega^r[U] = \wedge^r \Omega[U]$. □

20.4. **Rational differential forms.** Let X = irreducible smooth quasiprojective variety and $\omega \in \Omega^r[X]$.

Lemma. $\{p \in X : \omega(p) = 0 \in \wedge^r T^*_{X,p}\}$ *is closed.*

Proof. WLOG X affine, smooth such that $\omega = \sum_{i_1<\cdots<i_r} g_{i_1,\dots,i_r} df_{i_1} \wedge \cdots \wedge df_{i_r}$ such that $df_1, \dots, df_n$ form a basis of $T^*_{X,p}$ for all $p \in X$. Then $V(\omega) = V(g_{i_1,\dots,i_r})_{i_1<\cdots<i_r}$ is closed in X. □

21. Rational differential forms

X = irred smooth quasiprojective variety.

$\omega \in \Omega^r[X]$.

Last time: $V(\omega) = \{p \in X : \omega(p) \in \wedge^r T^*_{X,p}$ is zero$\}$ is a closed subset of X. In particular, if $\omega \in \Omega^r[X]$ and $\omega|_U = 0$ then $\omega = 0$.

Define $\Omega^r(X)$ to be the set of equivalence classes of pairs (U, ω) where $\omega \in \Omega^r[U]$ and where $(U_1, \omega_1) \sim (U_2, \omega_2)$ iff $\exists V \subset U_1 \cup U_2$ open and nonempty, dense, such that $\omega_1|_V = \omega_2|_V$. An equivalence class will be denoted $\omega = \{(U, \omega_U)\}$ or often $\{(U, \omega)\}$ and called a *rational r–form* on X.

Given $\omega \in \Omega^r(X)$, if $\exists (U, \omega_U) \in \omega$ then we call ω *regular on* U. The domain of regularity of ω is defined to be $\cup_{(U,\omega_U)\in\omega} U = U_\omega$.

Note: if X and Y are birational then $\Omega^r(X) \cong \Omega^r(Y)$ as $k(X) = k(Y)$–vector spaces. To see this, take open $U \subset X, V \subset Y$ with $U \cong V$. Then $\Omega^r(U) \cong \Omega^r(V)$ as $k[U]$–modules.

Theorem. *$\Omega^r(X)$ is a vector space over $k(X)$ of dimension $\binom{n}{r}$ where $n = \dim X$.*

Proof. Choose $U \subset X$ open and nonempty such that

(1) $\Omega^r[U]$ is a free $k[U]$–module of rank $\binom{n}{r}$.

(2) $\exists u_1, \ldots, u_n \in k[U]$ such that $du_1, \ldots, du_n$ is a basis of $\Omega^1[U]$ over $k[U]$ (so $\{du_{i_1} \wedge \cdots \wedge du_{i_r}\}$ is a basis of $\Omega^r[U]$).

Let $\omega \in \Omega^r(X)$. ω is regular on $U' \subset U$. Therefore $\omega = \sum g_{i_1 i_2 \cdots i_r} du_{i_1} \wedge \cdots \wedge du_{i_r}$ for $g_{i_1 i_2 \cdots i_r} \in k[U'] \subset k(X)$. Linear independence of $du_{i_1} \wedge \cdots \wedge du_{i_r}$ over $k(X)$ is easy. □

Theorem. *If $u_1, \ldots, u_n$ is a separable transcendence basis of $k(X)$ over k then the forms $du_{i_1} \wedge \cdots \wedge du_{i_r}$ form a basis for $\Omega^r(X)$ as a $k(X)$–vector space.*

Recall: $u_1, \ldots, u_n$ is a separable transcendence basis of L over K if

- $u_1, \ldots, u_n \in L$ are algebraically independent over k.
- $k(u_1, u_2, \ldots, u_n) \subset L$ is a finite separable extension. (Separable means that if $a \in L$ then $\text{min.poly}_{k(u_1,\ldots,u_n)}(a)$ has nonzero derivative.)

Example. X = curve $\subset \mathbb{A}^2$ with equation $y^2 - f(x) = 0$. Then $X \to \mathbb{A}^1$, $(x, y) \mapsto x$. And $k(x) = k(\mathbb{A}^1) \subset \frac{k(x)[y]}{y^2 = f(x)} = k(X)$. This is separable if char$k \neq 2$, since $\frac{d}{dT}(T^2 - f(x)) \neq 0$. Here $\Omega(X) = k(X)dx$ (cf. $2y dy = f'(x) dx$, so $dy = \frac{f'(x)}{2y} dx$ if char$k \neq 2$).

Recall from Chapter I of Shafarevich: every irreducible variety is birational to a hypersurface.

$$k(X) = k(x_1, \ldots, x_{n-1})[x_n]/f(x_n; x_1, \ldots, x_{n-1})$$

The proof also shows that $\frac{\partial f}{\partial x_n} \neq 0$.

Proof. (of theorem.) WLOG $X \subset \mathbb{A}^N$. Let $v \in k(X)$. Then the exists $F(v, u_1, \dots, u_n) = 0, F \in k[T_), \dots, T_n]$, F separable in v. For $i = 1, \dots, n$, we can find such F_i such that $F_i(x_i, u_1, \dots, u_n) = 0$.

Therefore,

$$0 = \frac{\partial F_i}{\partial T_0} dx_i + \sum_{j=1}^{n} \frac{\partial F_i}{\partial T_j} du_j$$

and $\frac{\partial F_i}{\partial T_0} \neq 0$, so $dx_i \in \sum_j k(X) du_j$. Therefore, $du_1, \dots, du_n$ generate $\Omega^1(X)$ over $k(X)$ and so $du_{i_1} \wedge \dots \wedge du_{i_r}$ generate $\Omega^r(X)$ over $k(X)$. Correct dimension $\implies$ they are a basis. ☐

21.1. **Behaviour under regular and rational maps.** Let $\varphi : X \to Y$ be a regular map of varieties. Then φ induces maps

$$\varphi^* : \Phi[Y] \to \Phi[X]$$
$$\Phi^r[Y] \to \Phi^r[X]$$
$$\Omega^r[Y] \to \Omega^r[X]$$

(check), all called φ^*. Basic idea: $\varphi^*(f dg) = (\varphi^* f) d(\varphi^* g)$ and $\varphi^*(f dg_1 \wedge \dots \wedge dg_r) = (\varphi^* f) d(\varphi^* g_1) \wedge \dots \wedge d(\varphi^* g_r)$.

Example. $\mathbb{A}^1 \to \mathbb{A}^2_{xy}$, $t \mapsto (t^2, t^3)$. $\varphi^* : \Omega^1[\mathbb{A}^2] \to \Omega^1[\mathbb{A}^1]$. Compute: $\varphi^*(dx) = d(\varphi^* x) = d(t^2) = 2tdt$ and $\varphi^*(dy) = 3t^2 dt$.

If $\varphi : X \dashrightarrow Y$ is dominant and rational and X, Y are irreducible then $\varphi^* : \Omega^r(Y) \to \Omega^r(X)$, where a $(U, \omega) \in \Omega^r(Y)$ is mapped to $(\varphi^{-1} U, \varphi^* \omega)$.

Theorem. *If X, Y are smooth and $k(X)$ has a separable transcendence basis over $k(Y)$ then $\varphi^* : \Omega^r(Y) \to \Omega^r(X)$ is an inclusion for all r.*

Cool Theorem. *Suppose $\varphi : X \dashrightarrow Y$ is dominant and X, Y smooth and irreducible, Y projective. Then*

$$\varphi^* \Omega^r[Y] \hookrightarrow \Omega^r[X].$$

Corollary. *If X, Y are birational and smooth and projective then $\Omega^r[X] \cong \Omega^r[Y]$ as k–vector spaces.*

(1) Cubics C_1, C_2 in $\mathbb{P}^2$, $C_1 \cap C_2 = \{p_1, p_2, \dots, p_9\}$. Let D be any cubic through $\{p_1, p_2, \dots, p_8\}$. The aim is to show that D also passes through p_9.

Let $i_1 : C_1 \hookrightarrow \mathbb{P}^2$ be the embedding. Then $i_1^*(C_2) = \sum_{i=1}^9 p_i$ and $i_1^*(D) = \sum_{i=1}^8 p_i + q$ for some q. Since $C_2 - D$ is a principal divisor on $\mathbb{P}^2$ (divide the equation of C_2 by equation of D), we get $0 = i_1^*(C_2 - D) = p_9 - q$. If $p_9 \neq q$, this implies that C is rational. So $p_9 = q$.

(2) Prove that there are four tangent lines to a plane cubic X that pass through a given point p (regarding $x \in T_{X,x}$ only if x is an inflexion point).

Consider m–torsion points, ie. x such that $\underbrace{x \oplus x \oplus \cdots \oplus x}_{m} = o$. There are exactly m^2 of these (proved earlier in chapter). We have $q \oplus q = \ominus p$ iff the tangent line to X at q passes through p. There exist x, y, z, o such that $x \oplus x = y \oplus y = z \oplus z = o$. The equations in the book for $\oplus$ etc. give that there exists a solution q to $q \oplus q = \ominus p$. The other solutions are then given by $q \oplus x, q \oplus y$ and $q \oplus z$.

Alternatively: $X \xrightarrow{\lambda} X$, $q \mapsto \ominus(q \oplus q)$ has $n(\lambda) = 4$ so degree 4.

(3) The exercise is to prove that there is a conic tangent to the cubic of the previous problem at x which passes through the points q with $x \in T_{X,q}$.

One solution is to take F to be the equation of the cubic in $\mathbb{P}^2$, and G to be the so-called *polar curve* of F, defined by $G = \alpha F_x + \beta F_y + \gamma F_z$ where $(\alpha : \beta : \gamma) = x$. Then $G(x) = 0$ by Euler's formula, $G(q) = 0$ for each of the given q by definition of these points being on the tangent lines, and G is tangent to X at x (these can all be checked by repeatedly applying Euler's identity for a homogeneous polynomial).

There may be another argument to solve this problem using Bézout's theorem.

(4) No.6. The exercise is to show that $\Omega^r[\mathbb{P}^n] = 0$.

Let $\mathbb{P}^n = \cup_{j=0}^n U_j$ and $t_i^j = x_i/x_j$. Then if $\omega \in \Omega^r[\mathbb{P}^n]$, then $\omega|_{U_j} = \sum g^j_{i_1 \cdots i_r} dt^j_{i_1} \wedge \cdots \wedge dt^j_{i_r}$. In $U_i \cap U_j$, $t^j_{i_k} = t^i_{i_k}/t^i_j$. Therefore, $dt^j_{i_k} = \frac{t^i_j dt^i_{i_k} - t^i_{i_k} dt^i_j}{(t^i_j)^2}$. So $\omega = \sum g^j_{\cdots} \frac{1}{(t^i_j)^{2r}} (t^i_j)^r dt^j_{i_1} \wedge \cdots \wedge dt^j_{i_r} - (\cdots)$. This equals $\sum \frac{\tilde{g}^j}{(t^i_j)^{2r}} dt^j_{i_1} \wedge \cdots \wedge dt^j_{i_r}$ for some polynomials $\tilde{g}$ in $t^j_0, \dots, t^j_n$. So $g^i_{\cdots}(t^i_0, \dots, t^i_n) = \frac{1}{(t^i_j)^{2r}} \tilde{g}(\frac{t^i_0}{t^j_0}, \dots, \frac{t^i_n}{t^i_j})$. Writing in terms of homogenised $\tilde{g}$ yields that all the $g^i_{i_1, \dots, i_r}$ are zero.

Could be made easier by using just two open sets U_0 and U_1, but would be the same calculation.

(5) This exercise shows that for a singular variety, Kähler differentials and $\Omega[X]$ may not be the same.

Let $A = k[X] = k[x,y]/(y^2 - x^3) = k[x] \oplus k[x]y$ as a $k[x]$–module. Assuming that $\Omega_A = (Adx + Ady)/A(2ydy - 3x^2dx)$, we claim that $3ydx - 2xdy \neq 0$ in Ω_A. Suppose it is. Then $3ydx - 2xdy = \alpha(2ydy - 3x^2dx)$ for some $\alpha \in A$. Therefore, $3y = -3\alpha x^2$, $-2x = 2\alpha y$ in A. But $y \neq 0$ in $A/(x^2)$, so this is impossible. However, $3ydx - 2xdy \in \Omega[X]$ is zero, because $y(3ydx - 2xdy) = 3y^2dx - 2xydy = 3y^2dx - 3x^3dx = 0$, so $3ydx - 2xdy$ is the zero function on $X \setminus V(y)$, while if $y = 0$ then $x = 0$ and so $3ydx - 2xdy = 0$.

We need to know also that if $A = k[x_1, x_2, \ldots, x_n]/(f_1, \ldots, f_r)$ then $\Omega_A = (Adx_1 + \cdots + Adx_n)/A(df_1, \ldots, df_r)$. This should be provable using the universal property of Ω_A.

23. The canonical class

$\varphi : X \dashrightarrow Y$ dominant, X, Y smooth irred.

Theorem. *If $k(X)$ has a separable transcendence basis over $k(Y)$ then $\varphi^* : \Omega^r(Y) \to \Omega^r(X)$ is an inclusion $\forall r$ (see book for definition of φ^*).*

Proof. Hypothesis: there exist $v_1, \ldots, v_s$ algebraically independent over $k(Y) \subset k(X)$ (via φ^*) such that $k(Y)(v_1, \ldots, v_s) \subset k(X)$ is finite and separable. Let $u_1, \ldots, u_t$ be a separable transcendence basis for $k(Y)$ over k. Let $\omega \in \Omega^r(Y)$. Then $\omega = \sum g_{i_1,\ldots,i_r} du_{i_1} \wedge \cdots \wedge du_{i_r}$. Then $\varphi^*\omega = \sum \varphi^* g_{i_1,\ldots,i_r} d(\varphi^* u_{i_1}) \wedge \cdots \wedge d(\varphi^* u_{i_r})$. Since the $\varphi^* u_i = u_i$ are part of a transcendence basis of $k(X)$ over k, $\varphi^*\omega = 0$ iff $\varphi^* g_{i_1,\ldots,i_r} = 0$ for all $i_1, \ldots, i_r$. This holds if and only if $g_{i_1,\ldots,i_r} = 0$ since the pullback of functions under φ is $1-1$ (φ is dominant). This in turn holds if and only if $\omega = 0$. □

Theorem. *Suppose $\varphi : X \to Y$ is dominant rational map, X, Y smooth and irreducible, Y projective. Then $\varphi^*(\Omega^r[Y]) \subset \Omega^r[X]$.*

Proof. There exists $Z \subset X$ closed with $\operatorname{co\,dim}_X Z \geq 2$ (uses projectivity of Y and smoothness) such that $\varphi : X \setminus Z \to Y$ is regular. Suppose $\omega \in \Omega^r[Y]$. There exists an open cover $\{U\}$ of X such that for each U, we may write $\varphi^*\omega = \sum g_{i_1,\cdots,i_r} du_{i_1} \wedge \cdots \wedge du_{i_r}$ where $g_{i_1,\ldots,i_r} \in k(X)$ are regular on $U \setminus Z$ and $du_{i_1}, \ldots, du_{i_r}$ are a basis of $\Omega[U]$ over $k[U]$ (so that $du_{i_1} \wedge \cdots \wedge du_{i_r}$ form a basis of $\Omega^r[U]$ over $k[U]$). But $g_{i_1,\ldots,i_r}$ is regular on $U \setminus (U \cap Z)$, and $\operatorname{co\,dim}_U(U \cap Z) \geq 2$. So $g_{i_1,\ldots,i_r}$ is actually regular on U. *(Note to self: have previously seen this last part of the argument referred to as Hartog's Theorem.)* □

Corollary. *If X, Y are smooth projective varieties and X, Y are birational then $\Omega^r[X] \cong \Omega^r[Y]$.*

Definition. *If X is a smooth projective curve, define the* genus $g = g(X) = \dim_k \Omega^1[X]$, *a.k.a.* $h^0(\Omega^1_X) = h^1(\mathcal{O}_X) = h^1$.

We know $g(\mathbb{P}^1) = 0$.

23.1. **Canonical class.** Let X = smooth of dimension n. Consider $\Omega^n[X]$ and $\Omega^n(X)$. What data is needed to give an element $\omega \in \Omega^n(X)$, $\omega \neq 0$? What we need: there exists an open cover $\{U_i\}$ and, on U_i, functions $u_1^{(i)}, \ldots, u_n^{(i)} \in k[U_i]$, $g^{(i)} \in k(X)^*$ such that $\omega = g^{(i)} du_1^{(i)} \wedge \cdots \wedge du_n^{(i)}$ on U_i, (ie. this (U_i, ω) is in the class ω). On $U_i \cap U_j$, $g^{(j)} = g^{(i)} J\left(\frac{u_1^{(i)},\ldots,u_n^{(i)}}{u_1^{(j)},\ldots,u_n^{(j)}}\right)$ where $J\left(\frac{u_1^{(i)},\ldots,u_n^{(i)}}{u_1^{(j)},\ldots,u_n^{(j)}}\right)$ is the Jacobian, which is regular and nonzero on $U_i \cap U_j$. Therefore, $\{(U_i, g^{(i)})\}$ is a Cartier divisor on X. Called $\operatorname{div}(\omega)$. Note:

(1) $\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega)$, $f \in k(X)^*$.
(2) $\operatorname{div}(\omega) \geq 0 \iff$ each $g^{(i)}$ is regular on $U_i \iff \omega \in \Omega^n[X]$.
(3) $\Omega^n(X)$ is a one-dimensional vector space over $k(X)$. If $\omega_1 \in \Omega^n(X) \setminus \{0\}$ then $\exists f \in k(X)^*$ such that $\omega_1 = f\omega$. Therefore, $\operatorname{div}(\omega_1) \sim \operatorname{div}(\omega)$.

Definition. $K_X \in \operatorname{Cl}(X)$ *is the class* $[\operatorname{div}(\omega)]$ *for any* $\omega \in \Omega^n(X)$. *Called the* canonical class *of* X.

Note: $\Omega^n[X] \cong L(\operatorname{div}(\omega)) \cong L(K_X)$. Therefore, for X smooth projective curve, $g(X) = \dim\Omega^1[X] < \infty$ since $\dim L(K_X) < \infty$.

Examples.

- $X = \mathbb{P}^n$, K_X.
- $X = V(F) \subset \mathbb{P}^n$, a smooth hypersurface. Compute K_X.
- X hyperelliptic curve.
- Invariant forms under a group.

$X = \mathbb{P}^n_{x_0 x_1 \dots x_n}$. Find K_X. On U_0, coordinates $t_i = x_i/x_0$, $1 \le i \le n$. Can take anything we want, so take $\omega = dt_1 \wedge \cdots \wedge dt_n$. Then $\operatorname{div}(\omega)|_{U_0} = 0$. So $\operatorname{div}(\omega) = \ell \cdot V(x_0)$ for some $\ell \in \mathbb{Z}$. On U_1, $t_1 = 1/u_1$, $t_2 = u_2/u_1$, …, $t_n = u_n/u_1$ where $u_1 = x_0/x_1, u_2 = x_2/x_1, \dots, u_n = x_n/x_1$ are coordinates on U_1. So $\omega = d(\frac{1}{u_1}) \wedge d(\frac{u_2}{u_1}) \wedge \cdots \wedge d(\frac{u_n}{u_1})$. Replacing $d(\frac{u_i}{u_1})$ by $(u_i du_1 - u_1 du_i)/u_1^2$, this becomes

$$\frac{-1}{u_1^{n+1}} du_1 \wedge du_2 \wedge \cdots \wedge du_n.$$

Therefore, $\operatorname{div}(\omega)|_{U_1} = -(n+1)V(x_0)$. So overall, $\operatorname{div}(\omega) = -(n+1)V(x_0)$, where $V(x_0)$ is the hyperplane section.

24. Hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$, $X = V(F)$ smooth hypersurface of degree m ($m = \deg f$). Let's compute

(1) K_X.
(2) $\Omega^n[X] = L(K_X)$.

Plan: to mimic the $\mathbb{P}^{n+1}$ case.

Let $U = U_0 = \mathbb{P}^{n+1} \backslash V(x_0)$. X is defined by the inhomogeneous equation $G(y_1, \dots, y_{n+1}) = 0$, $y_i = x_i/x_0$, $i = 1, 2, \dots, n+1$. Define $U_i = X \cap U_0 \setminus V(\partial G/\partial y_i)$ for $i = 1, \dots, n+1$. Note that the U_i cover $X \cap U_0$ (by smoothness) and $\Omega^n[U_i]$ is a free $k[U_i]$–module of rank 1 generated by $dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_{n+1}$ *(I believe this follows from the fact that the y_k with $k \neq i$ are local coordinates on U_i; dy_i can be written as a sum of the others because $\partial G/\partial y_i \neq 0$ and since the cotangent space at each point is n–dimensional, we get that $dy_1, \dots, \widehat{dy_i}, \dots, dy_{n+1}$ must be a basis for this space.)* Take

$$\omega_i = \frac{(-1)^i}{(\partial G/\partial y_i)} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_{n+1}$$

Claim. On $U_i \cap U_j$, $\omega_i = \omega_j$. If so, then $\omega = \{(\omega_i, U_i)\}$ defines an element of $\Omega^n[U \cap X]$.

Proof. $\sum_{i=1}^{n+1} \frac{\partial G}{\partial y_i} dy_i = 0$. Now wedge with $dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \widehat{dy_j} \wedge \cdots \wedge dy_{n+1}$, so

$$0 = \frac{\partial G}{\partial y_i} dy_i \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_{n+1} + \frac{\partial G}{\partial y_j} dy_j \wedge (\cdots)$$

then divide by $\frac{\partial G}{\partial y_i}\frac{\partial G}{\partial y_j}$ and get $0 = \omega_i - \omega_j$. □

Now compute $\operatorname{div}(\omega)$. From definition of ω, $\operatorname{div}(\omega)|_{X\cap U} = 0$. Therefore, $\operatorname{div}(\omega) =$ a sum with multiplicity of components of $V(x_0) \cap X$. Let $V = \mathbb{P}^{n+1} \setminus V(x_1)$.

Coordinates $z_1 = 1/y_1$, $z_i = y_i/y_1$, $2 \le i \le n+1$. So $y_1 = 1/z_1$, $y_i = z_i/z_1$. As before, $dy_1 = (-1/z_1^2)dz_1$ and $dy_i = (1/z_1)dz_i - (z_i/z_1^2)dz_1$. Plug into one of our ω_i, say ω_{n+1}. On $V \cap U_{n+1}$,

$$\omega = \left(\frac{-1}{z_1}\right)^{n+1} \frac{(-1)^{n+1}}{(\partial G/\partial y_{n+1})} dz_1 \wedge \cdots \wedge dz_n.$$

Let $H(z_1, \ldots, z_{n+1}) = F(z_1, 1, z_2, \ldots, z_{n+1})$. Then $H(z_1, \ldots, z_{n+1}) = z_1^m G(\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{n+1}}{z_1})$. Then

$$\frac{\partial H}{\partial z_{n+1}} = z_1^m \frac{\partial G}{\partial y_{n+1}} \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{n+1}}{z_1}\right) \frac{1}{z_1}$$

(by the chain rule). So

$$\omega = \left(\frac{-1}{z_1}\right)^{n+1} \frac{(-1)^{n+1}}{\left(\frac{1}{z_1^{m-1}} \frac{\partial H}{\partial z_{n+1}}\right)} dz_1 \wedge \cdots \wedge dz_n.$$

We already worked out $\Omega^n[U_i]$ *and hence* $\Omega^n[U \cap X]$. *A similar calculation yields* $\Omega^n[V \cap X] = k[V \cap X] \frac{1}{\partial H/\partial z_{n+1}} dz_1 \wedge \cdots \wedge dz_n$. We therefore get $\operatorname{div}(\omega)|_{V\cap U_{n+1}} = (m-n-2)H$ where $H = V(z_1) \cap U \cap V_{n+1}$. Therefore overall, $\operatorname{div}(\omega) = (m-n-2)H$, $H =$ class of a hyperplane section. So $K_X \sim (m-n-2)H$.

$L(K_X) \cong \Omega^n[X] \cong$ span of homogeneous polynomials of degree $m-n-2$ via $f \mapsto f\omega$. *(Every homogeneous polynomial* Φ *of degree* $m-n-2$ *gives* Φ/x_0^{m-n-2} *which belongs to* $L((m-n-2)V(x_0))$. *Conversely, any function in* $L((m-n-2)V(x_0))$ *is of this form. This takes a little bit of proving. See Shafarevich, III, 1.5.)*

Example. $X =$ smooth plane curve of degree m. $K_X \sim (m-3)H$ ($n = \dim X = 1$). Therefore, $L(K_X)$ has dimension $\ell(K_X) = \frac{(m-1)(m-2)}{2}$. If $m = 1$ or $m = 2$ then $\ell(K_X) = 0$. If $m = 3$, $K_X \sim 0$ and $\ell(K_X) = 1$. We found this differential form in one specific case. Also, $g(X) = 1$ (proves that X is not rational). If $m = 4$, then $K_X \sim H$, the hyperplane section. Called a canonical curve. Here $\ell(K_X) = 3$.

In $\mathbb{P}^3$, $X \subset \mathbb{P}^3$ of degree m. Then $K_X \sim (m-4)H$. If $m = 4$ and X is smooth, X is called a $K3$–surface. If $m = 3$, X is a cubic surface and $K_X \sim -H$ so $H \sim -K$, anticanonical embedding. Surfaces with $H \sim -K$ are called del Pezzo surfaces.

Question: Suppose $X \subset \mathbb{P}^4$ has degree m. Then $K_X \sim (m-5)H$. If $m = 3$ then cubic threefold, $K_X \sim -2H$. So $\Omega^3[X] = 0$. And $\Omega^3[\mathbb{P}^3] = 0$. So is X birational to $\mathbb{P}^3$? ie. is X rational?

Answer: No. (Clemens, Griffiths).

25. Hyperelliptic curves

Consider the equation

$$y^2 = F(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{2g+1})$$

where $F(X)$ has odd degree $n = 2g+1$ and no multiple roots. Let $Y = V(y^2 - F(X)) \subset \mathbb{A}^2$. Let $\overline{Y}$ be the projective closure of Y in $\mathbb{P}^2$. Then a calculation shows that $\overline{Y}$ is singular at $\infty = (0:1:0)$ (check). Let $X =$ normalisation of $\overline{Y}$. So X is a smooth projective curve and $\sigma : X \dashrightarrow \overline{Y}$ is birational and $\sigma : \sigma^{-1}(Y) \to Y$ is an isomorphism. Our plan:

- Understand X somewhat.
- Compute K_X and a basis for $\Omega^1[X]$.

First,

$$Y \to \mathbb{A}^1$$
$$(x, y) \mapsto x$$

is a rational map. Since Y is an open subset of X, this induces a rational map $f : X \dashrightarrow \mathbb{A}^1$. Since X is smooth, this is the same as a regular map $f : X \to \mathbb{P}^1$.

What is $\deg f$?

$X = Y \cup \{\text{bunch of points}\}$ where $Y \mapsto \mathbb{A}^1$ and the points map to ∞. If $\beta \in \mathbb{A}^1$ then $f^*(\beta) = ?$.

$$f^{-1}(\beta) = \begin{cases} 2 \quad \text{pts. if } F(\beta) \neq 0 \\ 1 \quad \text{pt. if } F(\beta) = 0 \end{cases}$$

Therefore, $f^*(\beta) = z' + z''$ or $2z$, where $z', z'' = (\beta, \pm\sqrt{F(\beta)})$ or $z = (\beta, 0)$. So $\deg f = 2$.

Know $X = Y \cup f^{-1}(\infty)$. What are $f^*(\infty), f^{-1}(\infty)$? $\deg f^*(\infty) = 2$. Therefore, $f^*(\infty) = z' + z''$ or $2z$. Let $u = 1/x$, a local parameter on $\mathbb{P}^1$ at ∞. In the first case, $\nu_{z'}(u) = 1$ and $\nu_{z''}(u) = 1$. So the rational function x has a pole of order one at z' and z''. Therefore, $\nu_{z'}(F(x)) = -(2g+1) = \nu_{z'}(y^2) = 2\nu_{z'}(y)$, a contradiction. Therefore, $f^*(\infty) = 2z_\infty$ for some point z_∞.

As a set, $X = Y \cup \{z_\infty\}$. We have the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{\sigma} & \overline{Y} & \subset & \mathbb{P}^2 \\ \downarrow f & & \cup & & \\ \mathbb{P}^1 & & Y & = & V(y^2 - F(x)) \end{array}$$

Let $p_i = \sigma^{-1}(\alpha_i, 0)$, $1 \leq i \leq 2g+1$. Then $z_\infty, p_1, \ldots, p_{2g+1}$ are exactly the points where $f^*(t) = 2t$. WLOG let $\alpha_1 = 0$ (via a change of coordinates). Then (check this):

$$\operatorname{div}(x) = 2p_1 - 2z_\infty$$

$$\operatorname{div}(y) = p_1 + \cdots + p_{2g+1} - (2g+1)z_\infty$$

Note that in Y, on $y \neq 0$, x is a local coordinate. On $y = 0$, ie. at $p_1, \ldots, p_{2g+1}$, y is a local coordinate. ($2ydy = F'(x)dx$).

What about z_∞? $\nu_{z_\infty}(x) = -2$, $\nu_{z_\infty}(y) = -(2g+1)$, $\nu_{z_\infty}(t) = 1$ where $t = x^g/y$. And $\operatorname{div}(t) = z_\infty + 2gp_1 - (p_1 + p_2 + \cdots + p_{2g+1})$.

25.1. **Differentials.** Let's compute K_X. Choose $\omega = dx/y = 2y/F'(x)$. Then $\omega \in \Omega^1[Y]$, therefore $\operatorname{div}(\omega) = \ell z_\infty$ for some $\ell \in \mathbb{Z}$. In $\mathcal{O}_{X,z_\infty}$, we have $x = ut^{-2}, y = vt^{-(2g+1)}$ where u, v are units in $\mathcal{O}_{X,z_\infty}$. So $dx = (1/t^2)du - (2u/t^3)dt$. But note $du = hdt$ for some function $h \in \mathcal{O}_{X,z_\infty}$. So

$$\frac{dx}{y} = \frac{t^{2g+1}}{v}\left(\frac{du}{t^2} - \frac{2u}{t^3}dt\right) = \frac{t^{2g+1}}{v}\left(\frac{ht - 2u}{t^3}\right)dt$$

So $\nu_{z_\infty}(dx/y) = 2g-2$. So $\operatorname{div}(\omega) = (2g-2)z_\infty$. Therefore, $K_X \sim (2g-2)z_\infty$.

What is $\Omega^1[X]$?

Claim: $\Omega^1[X] = \operatorname{span}_k\{\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y}\}$.

Why? Any $\omega \in \Omega^1[X]$ is of the form $\omega = (g_1(x) + yg_2(x))\frac{dx}{y}$ for g_1, g_2 polys in x. Need ω regular at z_∞. Expressing in terms of t, we get

$$\omega = \left(g_1\left(\frac{u}{t^2}\right) + \frac{v}{t^{2g+1}}g_2\left(\frac{u}{t^2}\right)\right)\frac{t^{2g-2}}{v}(ht - 2u)dt.$$

So $g_2 = 0$, otherwise pole, and $\deg(g_1) \leq g - 1$. So the claim is proved. Have shown that for X hyperelliptic,

- $\deg K_X = 2g - 2$.
- $\ell(K_X) = g$.
- $g(X) = g$.

26. Riemann's inequality

X = smooth projective curve.

Theorem. *There exists a constant $\gamma = \gamma(X)$ such that*

$$\ell(D) \geq \deg D + 1 - \gamma \qquad (*)$$

for all $D \in \mathrm{Div}(X)$.

Remarks.

(1) For now, let $s(D) := \deg D - \ell(D) + 1$. The theorem says that $s(D) \leq \gamma$.

(2) Let p_i, $1 \leq i \leq d$, be points of X. Then $0 < p_1 < p_1 + p_2 \cdots < p_1 + p_2 + \cdots + p_d$ and so

$$L(0) \subset L(p_1) \subset L(p_1 + p_2) \subset \cdots \subset L(p_1 + \cdots + p_d)$$

and $\dim L(0) = 1$, $\dim L(p_1) \leq 2$, $\dim L(p_1 + p_2) \leq 3$ and so on. Note that the number of non-jump locations is $(d+1) - \dim L(p_1 + \cdots + p_d) = d - \ell(D) + 1 = s(D)$ where $D = \sum p_i$. (Here, we say there is a jump location at i if $L(p_1 + \cdots + p_{i-1}) \subsetneq L(p_1 + \cdots + p_{i-1} + p_i)$.)

(3) Suppose we have $D_1 \leq D_2$. Then $\ell(D_2) \leq \ell(D_1) + \deg(D_2 - D_1)$ (we saw this). So $\deg D_1 - \ell(D_1) \leq \deg D_2 - \ell(D_2)$ and so $s(D_1) \leq s(D_2)$. So
   (a) if $s(D_2) \leq \gamma$ then $s(D_1) \leq \gamma$.
   (b) if we show $(*)$ for all effective divisors D then $(*)$ holds for all divisors since any divisor $D = D_2 - D_1$, $D_2, D_1 \geq 0$. So $D \leq D_2$.

We want to show $s(D) = \deg D - \ell(D) + 1 \leq \gamma$.

Idea: let $t \in k(X)^*$ be such that $t : X \to \mathbb{P}^1$ has degree n (n can be anything - we just pick a rational function that is not constant). Let D_∞ = divisor of poles of t. Then we will show $\ell(rD_\infty) \geq rn - \gamma + 1$ for r sufficiently large.

Write $D_\infty = \sum n_i p_i$. Each $n_i > 0$, $n_i = -\nu_{p_i}(t)$, $\sum n_i = n$. Let $w_1, \dots, w_n$ be a basis of $k(X)$ over $k(t)$.

Consider w_i. Suppose its poles outside $\text{supp}D_\infty$ are $q_1, \dots, q_\ell$ with orders $k_1, \dots, k_\ell$. Consider $u_i := w_i \prod_{j=1}^{\ell}(t - t(q_j))^{k_j}$. This has no poles outside $\text{supp}D_\infty$. So $\{u_1, \dots, u_n\}$ is a basis of $k(X)$ over $k(t)$ and $u_i \in L(ND_\infty)$ for N sufficiently large, for all $1 \leq i \leq n$.

Let $\nu_{p_i}(u_j) = -m_{ij}$ (these can be negative). Then if $g(t)$ is a polynomial in t of degree k, then $\nu_{p_i}(g(t)u_j) = -kn_i - m_{ij}$. So $g(t)u_j \in L(rD_\infty)$ if and only if $kn_i + m_{ij} \leq rn_i$ for all i (for a fixed j). Let $m = \max_{i,j}(m_{ij}/n_i)$. Then $k \leq r - m \iff k + m \leq r \Rightarrow g(t)u_j \in L(rD_\infty)$. So $\ell(rD_\infty) \geq n(r - m + 1) = rn - mn - n$ (since we can choose $g(t) = 1, t, \dots, t^{r-m}$). Let $\gamma = n(m-1)+1$. Then $s(rD_\infty) \leq \gamma$ for $r >> 0$. But then, true for all r by earlier remarks.

Finally, $D = \sum_{i=1}^{\ell} k_i q_i$. If $q_i \notin \text{supp}D_\infty$, set $u = \prod_{q_i \notin \text{supp}D_\infty}(t - t(q_i))^{k_i}$. Then $\text{div}(u) = \sum_{\substack{q_i \notin \text{supp}D_\infty \\ a_i > 0}} -$(something supported on D_∞). Therefore, if $D' = D - \text{div}(u) =$ (negative off D_∞)+ (positive on D_∞), so $D' < rD_\infty$ for some $r >> 0$. So $s(D') \leq s(rD_\infty) \leq \gamma$. And $s(D) = s(D')$ since $D \sim D'$. This completes the proof.

27. The Riemann-Roch theorem

X smooth projective curve, K = canonical class of X, g = genus of X. Then

Theorem (Riemann-Roch)**.** *For any divisor D on X*

$$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

27.1. Simple remarks and consequences.

(1) $D = 0$; $1 - \ell(K) = \ell(D) - \ell(K) = 0 + 1 - g$. So $\ell(K) = g$ (this was our definition of g).

(2) $D = K \Rightarrow \ell(K) - 1 = \deg K + 1 - g \Rightarrow \deg K = 2g - 2$.

(3) If $\deg D \leq 2g - 1$ then $\ell(K - D) = 0$ and therefore $\ell(D) = \deg D + 1 - g$.

(4) Recall: if $p \in X$ then $X \cong \mathbb{P}^1$ iff $\ell(p) = 2$.

If $g = 0$ then $\deg K = 2g - 2 = -2$. So $\ell(p) - \ell(K - p) = \ell(p) - 0 = 1 + 1 - 0 = 2$. So $g = 0 \Rightarrow X \cong \mathbb{P}^1$.

(5) $g = 1 \Rightarrow \deg K = 0$, $\ell(K) = 1$. If $D > 0$ then $\ell(D) = \deg D$. We showed that this implies $X \cong$ smooth plane cubic curve.

(6) $g = 2 \Rightarrow \deg K = 2$ and $\ell(K) = 2$.

Corollary. *Let $f_0, \ldots, f_n$ be a basis of $L(D)$ and assume that D is effective. Let $\varphi_{|D|} : X \to \mathbb{P}^n$ defined by $p \mapsto (f_0(p) : f_1(p) : \cdots : f_n(p))$ be the corresponding rational map (and it is therefore regular in this case). Suppose*

(1) *$\ell(D - p) = \ell(D) - 1$ for all $p \in X$.*

(2) *$\ell(D - p - q) = \ell(D) - p - q$ for all $p, q \in X$ (including $p = q$).*

Then $\varphi_{|D|}$ is an embedding.

Note: if $\ell(D - p - q) = \ell(D - p)$ and $p \neq q$ then $\varphi_{|D|}(p) = \varphi_{|D|}(q)$. If $\ell(D - 2p) = \ell(D - p)$ then φ fails to be an isomorphism on tangent spaces.

Corollary. *If $\deg D \geq 2g + 1$ then $\varphi_{|D|}$ is an embedding $X \hookrightarrow \mathbb{P}^n$, $n = \ell(D) - 1$.*

Proof. If E has degree $\geq 2g - 1$ then $\ell(E) = \deg E + 1 - g$. So $\ell(D) = \deg D + 1 - g$, $\ell(D - p) = \deg D - 1 + 1 - g = \ell(D) - 1$ and $\ell(D - p - q) = \deg D - 2 + 1 - g = \ell(D) - 2$. □

Example. $g = 2$. Let $\deg D = 5$. Then $\ell(D) = 5 + 1 - 2 = 4$. So $X \hookrightarrow \mathbb{P}^3$ and X is isomorphic to a curve of degree 5 in $\mathbb{P}^3$. *(Note to self: the reason why X has degree 5 is that the pullback of the hyperplane section under the map φ associated to D is equivalent to D.)*

Question: when is $\varphi_{|K|}$ an embedding?

$g = 0, 1$: no, because $\ell(K) = g$. If $g \geq 2$, need $\ell(K - p) = \ell(K) - 1$ and $\ell(K - p - q) = \ell(K) - 2$ for all $p, q \in X$. RR says that $\ell(p) - \ell(K - p) = 1 + 1 - g$. Know $\ell(p) = 1$, so $\ell(K - p) = g - 1 = \ell(K) - 1$. Now let $D = p + q$. RR says $\ell(p + q) - \ell(K - p - q) = 2 + 1 - g$, so $\ell(K - p - q) = g - 3 + \ell(p + q)$. For this to be equal to $g - 2$, we need $\ell(p + q) = 1$. We know $\ell(p + 1) = 1$ or 2, since $\ell(p)$ can't increase by more than 1 when you add a point. If $\ell(p + q) = 1$ for all $p, q \in X$, then $\varphi_{|K|}$ is an embedding. Suppose $\ell(p + q) = 2$. Then let $f \in L(p + q)$ be nonconstant. Therefore, $f : X \to \mathbb{P}^1$, a $2 - 1$ map since $\text{div}_infty(f) = p + q$.

Definition. *X is called* hyperelliptic *if $\exists f : X \to \mathbb{P}^1$ with* $\deg(f) = 2$.

X is called trigonal *if $\exists f : X \to \mathbb{P}^1$ with* $\deg(f) = 3$.

Corollary. *$\varphi_{|K|}$ is an embedding iff X is not hyperelliptic.*

Example. *Let $g(X) = 2$. Then X is hyperelliptic;* $\deg K = 2$, $\ell(K) = 2$.

28. Schemes

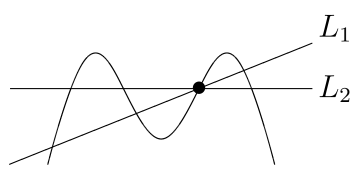
Notation: $h^0(D) = \ell(D)$.

RR: $\ell(D) - \ell(K - D) = \deg D + 1 - g$.

If $\exists X \overset{2-1}{\to} \mathbb{P}^1$ and $g \geq 2$, X is called hyperelliptic. If $\exists X \overset{3-1}{\to} \mathbb{P}^1$ and $g \geq 3$, X is called trigonal.

If $g \geq 2$, X is not hyperelliptic if and only if $\varphi_{|K|} : X \to \mathbb{P}^{g-1}$ is an embedding. Consider $g = 3$. Suppose X is not hyperelliptic. Then $\varphi_{|K|} : X \hookrightarrow \mathbb{P}^2$ is an isomorphism $X \hookrightarrow C \subset \mathbb{P}^2$ with $\deg C = \deg K = 2g - 2 = 4$. Therefore, if $g = 3$ and X is not hyperelliptic, then X is isomorphic to a smooth plane quartic.

Conversely, suppose $X \subset \mathbb{P}^2$ is a smooth plane quartic. Then $g(X) = 3$ and also, if $H =$ hyperplane section of $X =$ line $\cap X$, then $K \sim (4 - 3)H = H$. So $\varphi_K = \varphi_H = \text{id}$, an isomorphism. So X cannot be hyperelliptic. Therefore, $g = 3$, not hyperelliptic $\iff$ smooth plane quartic. Such an X is trigonal because if L_1 and L_2 are distinct lines passing through a point and they both intersect X in exactly four points, then $f = L_1/L_2$ is a degree 3 map to $\mathbb{P}^1$ (using that $\deg f^*(q) = \deg(f)$). Picture:



28.1. **Schemes and sheaves.** General plan: first generalise the notion of affine variety. $A = k[x_1, \ldots, x_n]/I$, $k = \overline{k}$, I radical ideal, then $A = k[X]$, $X = V(I) \subset \mathbb{A}^n$.

To allow:

(1) k not alg. closed, or not even a field, eg. $k = \mathbb{Z}$.
(2) A has nilpotents, eg. $A = k[x]/(x^2)$.
(3) A might not be f.g. over k or $\mathbb{Z}$, eg. A_p, A_f.

In fact, let's allow any A which is commutative with unity. We will define Spec$A$

- as a set.

- give it a topology.
- scheme structure.

$$\text{Spec}A = \{P \subset A | P \text{ is a prime ideal}\}$$

note: $(1) = A$ is not a prime ideal.

Example. $X \subset \mathbb{A}^n$ affine variety. $p \in X \overset{1:1}{\longleftrightarrow} m_p \subset A$ maximal ideal. $A = k[X]$. Let $\text{maxspec}A = \{m \subset A | m \text{ maximal}\}$. Recall that a regular map $X \to Y$, $k[Y] = B$, corresponds to a ring homomorphism $B \to A$. We want that if $B \to A$ is a ring homomorphism, would like a map $\text{maxspec}A \to \text{maxspec}B$. This doesn't work for general A and B, eg. $\mathbb{Z} \to \mathbb{Q}$,

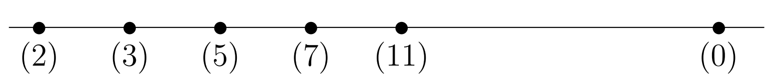
$$\text{maxspec}(\mathbb{Q}) \to \text{maxspec}(\mathbb{Z})$$
$$\{(0)\} \to \{(2), (3), (5), \ldots\}$$

No such natural map exists. Idea: use Spec$A$ instead.

Definition. *Let $f : A \to B$ be a map of rings. Then define the associated map $a_f : \text{Spec}B \to \text{Spec}A$ via $a_f(p) = f^{-1}(p)$.*

Check that $f^{-1}(p)$ is a prime ideal.

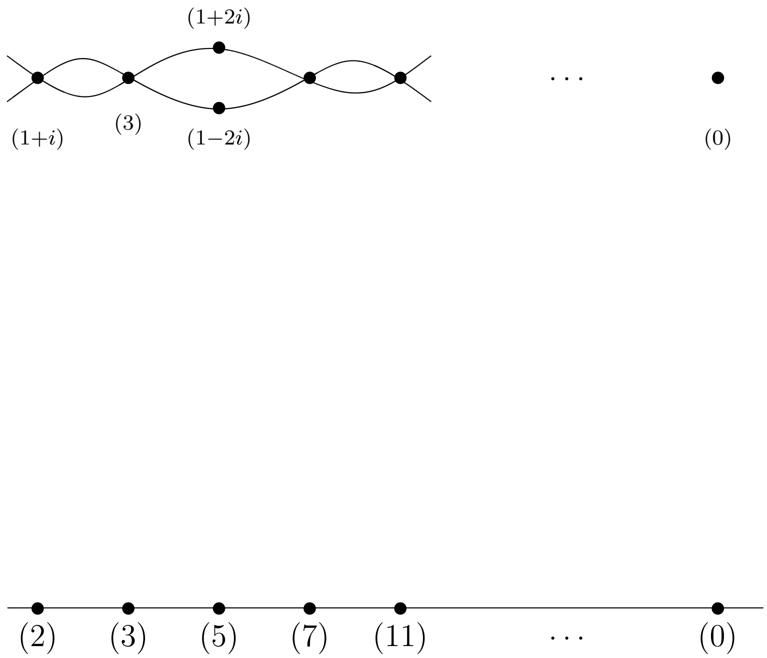
Example. $\mathbb{Z} \to \mathbb{Z}[i] = \mathbb{Z}[X]/(X^2 + 1)$. Let's "understand" the map $\text{Spec}\mathbb{Z}[i] \to \text{Spec}\mathbb{Z}$. Spec$\mathbb{Z}$ looks like:



What about Spec$\mathbb{Z}[i]$?

0 is a prime ideal. If $a + bi \in Q \in \text{Spec}\mathbb{Z}[i]$ then $a^2 + b^2 \in Q$. So there is a prime ideal $p \in \mathbb{Z}$ with p prime, $p \in Q$. Recall: $\mathbb{Z}[i]$ is a PID. If $p \equiv 1 \pmod 4$ then there are 2 primes of $\mathbb{Z}[i]$ containing p. If $p \equiv 3 \pmod 4$ then $p\mathbb{Z}[i]$ is prime. If $p = 2$ then the only possible

prime of $\mathbb{Z}[i]$ containing p is $(1+i) = (1-i)$. The picture is:



Example. $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. Spec$\mathbb{Z}[x] =$? $Q \subset \mathbb{Z}[x]$ prime. Then either:

(1) $Q = 0$.

(2) $Q = (f(x))$, f irreducible over $\mathbb{Z}$.

(3) $Q = (p)$, $p \in \mathbb{Z}$ prime.

(4) $Q = (p, f(x))$, $\overline{f} \in (\mathbb{Z}/p\mathbb{Z})[x]$ irreducible and f irreducible.

Get a picture a bit like the previous one, except that there are many points over each point of Spec$\mathbb{Z}$. For example, over zero have (0) and $(f(x))$ for all irreducible f.

Other good examples: $A \to A/I$, $A \to A_p$, $A \to A_f$.

29. Local properties

Example. $X = \text{Spec}\mathbb{C}[x, y]$. What are the points of this?

(1) $p = (0)$.

(2) $p = (f(x, y))$, $f(x, y)$ irred. over $\mathbb{C}$.

(3) $m_p = (x - p_1, y - p_2)$, $p = (p_1, p_2) \in \mathbb{C}^2$.

The first two kinds are known as generic points.

Examples. (1) $A \xrightarrow{\pi} A/I$, $\text{Spec}(A/I) \to \text{Spec}A, p \mapsto \pi^{-1}(p)$. Recall $\text{Spec}(A/I) = \{p \in \text{Spec}A : p \supset I\}$. The map is the inclusion.

(2) $A \to A_f$, $f \in A$. Then get $\text{Spec}A_f = \{p \in \text{Spec}A : f \notin p\} \to \text{Spec}A$, again an inclusion.

(3) $A \to A_S = S^{-1}A$, $S \subset A$ multiplicative set. ($A_p = (A \setminus p)^{-1}A$ is a special case.) So get $\text{Spec}(A_S) \to \text{Spec}A$, where $\text{Spec}(A_S) = \{p \in \text{Spec}A : p \cap S = \varnothing\}$.

29.1. **"Points" and functions of** Spec$A$**.**

Definition. *If* $p \in \text{Spec}A$, $k(p) = (A/p)_0 = A_p/pA_p$ *is a field. Called the* residue field *of* A *at* p.

We get a map $A \xrightarrow{\text{ev}_p} k(p)$. Kernel is p. (Note $\text{Spec}(k(p)) \to \text{Spec}A$ is $\{\text{pt.}\} \mapsto p$.) Suppose $f \in A$. This defines a "function" for $p \in \text{Spec}A$, $f(p) = \text{ev}_p(f)$. (Note: if $p = (x_1 - a_1, \ldots, x_n - a_n) \subset A = k[x_1, \ldots, x_n]/I$, then $k(p) = k$. And $A \to k = k(p)$ is the map $f \mapsto f(a_1, \ldots, a_n)$.)

Example. $f \in \mathbb{Z}$. f defines a function as above. $f(0) = f \in \mathbb{Q}$. $f(p) = f \pmod{p} \in \mathbb{Z}/p\mathbb{Z}$ (a field) for p prime.

Question: Does the function determine the element of A? ie. suppose $f, g \in A$ and $f(p) = g(p)$ for all $p \in \text{Spec}A$. Is $f = g$ in A?

$f(p) = g(p)$ iff $f - g \in p$, so $f(p) = g(p)$ for all $p \in \text{Spec}A$ iff $f - g \in \bigcap_{p \in \text{Spec}A} p$. Recall: $\bigcap_{p \in \text{Spec}A} p = \sqrt{(0)} = \{f \in A : f^n = 0 \text{ for some } n\}$. So $f(p) = g(p)$ for all p iff $f - g$ is nilpotent.

If $\sqrt{0} = 0$, ie. A has no nonzero nilpotent elements, then answer to the question is yes. On the other hand, if $\sqrt{0} \neq 0$, the answer is no.

Example. $A = k[x]/(x^2)$. Then $\text{Spec}A = \{(x)\}$ (since any prime ideal contains x). Write $p = (x)$. Then for $a, b \in k$, $(a + bx)(p) = a \in k(p) = k$, since $x = 0$ in k. Here the function does not determine the element.

Key idea:

For X a quasiprojective variety, a lot of what we have done so far in Volume I depends only on the local ring $\mathcal{O}_{X,p}$.

(1) $\dim_p(X) =$ Krull dim. $\mathcal{O}_{X,p}$.

(2) $T_{X,p} = (m_{X,p}/m_{X,p}^2)^*$.

(3) X is smooth at p if $\dim_k(m_{X,P}/m_{X,p}^2) =$ Krull dim. $\mathcal{O}_{X,p}$.

Assume A Noetherian.

Definition. *If (A, m) is a local ring then A is a* regular local ring *if*

$$\dim_{A/m}(m/m^2) = \dim A$$

where $\dim A$ *means Krull dimension.*

Definition. *If $X = \mathrm{Spec}A$ then let, for $p \in \mathrm{Spec}A$, $\mathcal{O}_{X,p} = A_p$.*

Then:

- $\dim_p X := \dim(\mathcal{O}_{X,p}) = \dim A_p$.
- *The tangent space*

$$T_{X,p} := \mathrm{Hom}_{k(p)}(pA_p/p^2A_p, k(p)),$$

  where $k(p) = A_p/pA_p$. $T_{X,p}$ is a vector space over $k(p)$.
- *X is regular (smooth) at p if (A_p, pA_p) is a regular local ring.*

Exercises. (1) *Suppose that $A = k[x_1, \ldots, x_n]/I = k[X]$, $X \subset \mathbb{A}^n$. Suppose $Y \subset X$ is irreducible. So $Y = V(p)$, $p \subset A$ prime ideal. Show that A_p is regular $\iff$ $Y \not\subseteq$* $\mathrm{sing}(X)$ $\implies$ *for almost all $q \in Y$, q is smooth on X.*

(2) *Find all the points of $X = \mathrm{Spec}(\mathbb{Z}[mi])$ which are not regular. Here, $m \in \mathbb{Z}$.*

30. The Zariski topology

$X = \mathrm{Spec}A$. If $E \subset A$, $I = (E)$, define $V(E) = V(I) = \{p \in \mathrm{Spec}A | p \supset I\}$.

Easy facts:

(1) $V(I) \cup V(J) = V(I \cap J)$.

(2) $\bigcap_\alpha V(E_\alpha) = V(\bigcup E_\alpha)$.

(3) $\bigcap_\alpha V(I_\alpha) = V(\sum I_\alpha)$, $I_\alpha = (E_\alpha)$.

Definition. *The Zariski topology on $X = \mathrm{Spec}A$ is the topology whose closed sets are the $V(I)$.*

Remarks:

(1) $\varphi : A \to B$ gives ${}^a\varphi : \mathrm{Spec}A \to \mathrm{Spec}B$ continuous. Why? Because for $E \subset A$, $({}^a\varphi)^{-1}V(E) = V(\varphi(E))$ by a definition chase.

(2) $\varphi : A \to A/I$. Then the image of ${}^a\varphi : \mathrm{Spec}(A/I) \to \mathrm{Spec}A$ is the closed subset $V(I)$, and ${}^a\varphi$ is a homeomorphism onto its image.

(3) $\varphi : A \to A_S$ gives $\mathrm{Spec}(A_S) \to \mathrm{Spec}A$ and the image is the open set

$$\{p \in \mathrm{Spec}A | p \cap S = \varnothing\}.$$

Exercise: show that this set is open.

(4) $\varphi : A \to A_f$ gives ${}^a\varphi : \mathrm{Spec}A_f \to \mathrm{Spec}A$ with image U_f open.
$U_f = \{p \in \mathrm{Spec}A | f \notin p\}$. Also denoted D_f, $D(f)$, ... Equals $\mathrm{Spec}A \setminus V(f)$.

Definition. *If X is a topological space then $\mathcal{B} = \{U_f\}_{f\in A}$ is a* basis *of the topology on X if*

(1) *U_f open.*

(2) *If $U \subset X$ is open then $U = \bigcup_{U_f \subset U} U_f$.*

$\mathcal{B}$ is called nice *if $U_f \cap U_g \in \mathcal{B}$ for all $U_f, U_g \in \mathcal{B}$.*

Key example: $X = \mathrm{Spec}A$.

(1) $\{U_f : f \in A\}$ is a basis for the Zariski topology on X. Why? $U = X \backslash V(I) = \bigcup_{f\in I} U_f$. And if $I = (f_1, \dots, f_r)$ then $U = U_{f_1} \cap \cdots \cap U_{f_r}$.

(2) If $f_i \in A$, $i \in \Lambda$ then $\bigcup_{i\in\Lambda} U_{f_i} = X$ iff $(f_i)_{i\in\Lambda} = A$ because it can't be contained in a maximal ideal.

(3) $U_f \cap U_g = U_{fg}$ (since $V(fg) = V(f) \cup V(g)$) so $\mathcal{B} = \{U_f\}_{f\in A}$ is a nice basis.

(4) $\bigcup_{f_i\in\Lambda} U_{f_i} = X \setminus V(f_i : i \in \Lambda)$.

(5) When is $U_f \subset U_g$? $U_f \subset U_g$ iff for all p, $f \notin p \implies g \notin p$ iff for all p, $g \in p \implies f \in p \Leftrightarrow \overline{f} \in A/(g)$ is nilpotent $\Leftrightarrow \overline{f}^n = 0 \Leftrightarrow f^n = gu$ for some $n > 0$ and some $u \in A$. So $U_f \subset U_g \Leftrightarrow f^n = gu \Leftrightarrow f \in \sqrt{(g)}$.

(6) $U_f = \varnothing \Leftrightarrow V(f) = X$ so f nilpotent.

Proposition. $\mathrm{Spec}A$ *is compact, ie. every open cover has a finite subcover.*

Proof. Exercise. □

Example. Let $X = \mathrm{Spec}\mathbb{C}[x,y]$. Let $p \subset \mathbb{C}[x,y]$ be prime but not maximal. Then $\{p\} \subset X$. Closure of a point = ? $\overline{\{p\}} = V(p) = \{q \in \mathrm{Spec}\mathbb{C}[x,y] : q \supset p\}$. Therefore, p is a closed point $\iff p$ is maximal.

If $p, q \in X$, say q is a *specialization* of p if $q \in \overline{\{p\}}$. If $\overline{\{p\}} = X$, say p is a *generic point* of X.

Example. $\mathrm{Spec}\mathbb{C}[x,y,z]$. Here, $V(x,y,z)$ is a closed point. $V(x,y)$ is another point. Its closure is itself, plus all the points on the line $x = y = 0$. $V(z)$ is another point. Its closure is itself, together with the points corresponding to all subvarieties of $z = 0$.

30.1. **Irreducible decomposition.** $X = X_1 \cup X_2$, X_1, X_2 proper closed $\iff X$ reducible.

Proposition. *If A is Noetherian then* $\mathrm{Spec}A = X_1 \cup \cdots \cup X_r$*, X_i closed and irreducible. This decomposition is unique.*

Do as an exercise, or look in book. Similar to irreducible decomposition for affine varieties.

31. Presheaves

Basic idea: if X is an affine variety and $U \subset X$ is open then $\mathcal{O}_X(U)$ = ring of regular functions on U. If $U \subset V$ are open sets, we get a restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Definition. *Let X be a topological space. Given the following data:*

(1) *For every open set U, a set $\mathcal{F}(U)$.*
(2) *For every inclusion $U \subset V$ of open sets, a map (called restriction) $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$.*

This system of data $\mathcal{F}$ is called a presheaf *if*

(1) *$\mathcal{F}(\varnothing)$ = single element set.*
(2) *$\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ for all open U.*
(3) *For every inclusion of open sets $U \subset V \subset W$, $\rho_U^W = \rho_U^V \rho_V^W$.*

If all the sets $\mathcal{F}(U)$ are groups, rings, A–modules etc. and all the ρ_U^V are morphisms of such, then $\mathcal{F}$ is called a presheaf of groups, rings, etc.

Examples. (1) Let X and Y be topological spaces. Let $\mathcal{F}(U) = \{f : U \to Y : f \text{ continuous}\}$ and ρ_U^V the obvious restriction maps. Then $\mathcal{F}$ is a presheaf, denoted $C(X,Y)$.

(2) $X =$ differential manifold, $C^\infty(X)(U) = \{f : U \to \mathbb{R} : f \text{ is } C^\infty\}$.

(3) X irred. quasiprojective variety, $\mathcal{F}(U) =$ ring of rational functions of X which are regular on U.

31.1. **Key example/construction:** $X = \text{Spec}A$.

Define presheaf $\mathcal{O}_X$.

Simplest case: $A =$ domain, $K = \text{Frac}(A)$. Define

$$\mathcal{O}_X(U) := \{f \in K : \forall p \in U \text{ there exists an expression}$$
$$f = a/b \text{ with } a, b \in A \text{ and } b(p) \neq 0 (\text{ie. } b \notin p)\}$$

Note: $\mathcal{O}_X(U)$ is a subring of K for all U. If $U \subset V$ then $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is just the inclusion $\mathcal{O}_X(V) \subset \mathcal{O}_X(U) \subset K$. Also, $\mathcal{O}_X(\varnothing) := 0$ (zero counts as a ring). Then $\mathcal{O}_X$ is a presheaf of rings on X.

Proposition. *If A is a domain then $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X(U_f) = A_f$ for all f.*

Proof. Let $u \in K$ [Note that $A \subset \mathcal{O}_X(X) \subset K$; $A_f \subset \mathcal{O}_X(U_f) \subset K$]. Case 1: $\mathcal{O}_X(X)$. For all $p \in \text{Spec}A$, $u = a_p/b_p$ with $a_p, b_p \in A$, $b_p \notin p$. Let $I = (b_p : p \in \text{Spec}A)$. Then I cannot be contained in any prime ideal, and so $I = A$. So there exist $p_1, \dots, p_r \in \text{Spec}A$ and $c_1, \dots, c_r \in A$ such that $\sum c_i b_{p_i} = 1$. Then $b_{p_i} u = a_{p_i}$ implies $u = \sum c_i b_{p_i} u = \sum c_i a_{p_i} \in A$.

Case 2: $\mathcal{O}_X(\text{Spec}A_f)$, $I = (b_p : p \in \text{Spec}A_f)$ where $u = a_p/b_p$ for all p with $f \notin p$, and $b_p \notin p$. Then $IA_f = (1)$. So there are $p_1, \dots, p_r \in U_f$ and $c_i \in A$ such that $1 = \sum c_i b_{p_i}/f^n$ for some $n \geq 0$. So $f^n = \sum u_i b_{p_i}$. So $uf^n = \sum c_i a_{p_i}$ and $u \in A_f$. □

What about if A is not a domain? eg. $A = k[x]/(x^2)$ or worse. Want to define $\mathcal{O}_X$. Let's define

$$\mathcal{O}_X(U_f) = A_f$$

for any $f \in A$, so $\mathcal{O}_X(X) = A$.

If $f \in \sqrt{(0)}$ then $\mathcal{O}_X(U_f) = \mathcal{O}_X(\varnothing) = A_f = 0$. What if $U_f = U_g$? This is true iff $\text{rad}(f) = \text{rad}(g)$ which implies $A_f \cong A_g$.

Suppose $U_f \subset U_g$. Then $f^n = gu$ for some u and some n. Then there exists a natural map $A_g \to A_f$ given by $a/g^m \mapsto au^m/f^{mn}$. (Or as an exercise, can get this map naturally by using the definition of localisation via a universal property.)

So $\mathcal{O}_X$ would be a presheaf if we just considered open sets of the form U_f. Now we want to define $\mathcal{O}_X(U)$ in general. In the case $X = \mathrm{Spec}A$ and A a domain, then $\mathcal{O}_X(U) = \bigcap_{U_f \subset U} \mathcal{O}_X(U_f) \subset K$.

Suppose we're given a poset Λ and a bunch of sets E_α for $\alpha \in \Lambda$, and if $\alpha \leq \beta$ there is a map $f_\alpha^\beta : E_\beta \to E_\alpha$ such that

(1) $f_\alpha^\alpha = \mathrm{id}_{E_\alpha}$ for all α.

(2) If $\alpha \leq \beta \leq \gamma$ then $f_\alpha^\gamma = f_\alpha^\beta f_\beta^\gamma$.

Then define

$$\varprojlim E_\alpha := \{(x_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} E_\alpha : f_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta\}$$

Notes:

(1) There exist natural maps $\varprojlim E_\alpha \to E_\alpha$ for all α (the projections).

(2) If E_α are all groups, rings, A–modules, etc. then so is $\varprojlim E_\alpha$.

(3) Suppose $\alpha \leq \beta \implies E_\beta \subset E_\alpha \subset S$ for some set S. Then in fact $\varprojlim E_\alpha = \bigcap_\alpha E_\alpha$.

Definition. *A a ring, $X = \mathrm{Spec}A$, $\mathcal{O}_X(U_f) = A_f$ for all $f \in A$. For a general open $U \subset \mathrm{Spec}A$, define*

$$\mathcal{O}_X(U) := \varprojlim_{U_f \subset U} \mathcal{O}_X(U_f).$$

ie. the indexing set is $\Lambda = \{U_f \subset U\}$ under inclusion.

Exercise. *Check that $\mathcal{O}_X$ is a presheaf of rings.*

32. Sheaves

Suggested homework problems: (Shafarevich vol. II)

Sect. 1 p. 15 #3

Sect. 2 p. 25 # 2,6,7

Sect. 3 p. 39 # 1,2

32.1. **Sheaves.** A presheaf $\mathcal{F}$ on X is given by $\mathcal{F}(U)$ for all open U and restriction maps $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ for every inclusion $U \subset V$. Want to capture other properties of functions, eg.

(1) Locally 0 $\implies$ 0.
(2) Functions which agree on overlaps can be glued together.

Definition. *A presheaf $\mathcal{F}$ on X is a* sheaf *if for any open set $U \subset X$ and any open cover $U = \bigcup_{\alpha\in\Lambda} U_\alpha$ then the sheaf axiom holds:*

If $s_\alpha \in \mathcal{F}(U_\alpha)$ for all $\alpha \in \Lambda$ and $\rho^{U_\alpha}_{U_\alpha\cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(s_\beta) \in \mathcal{F}(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in \Lambda$ then there exists a unique $s \in \mathcal{F}(U)$ with $\rho^U_{U_\alpha}(s) = s_\alpha$ for all $\alpha \in \Lambda$.

Note: this implies that if $s_1, s_2 \in \mathcal{F}(U)$ and $\rho^U_{U_\alpha}(s_1) = \rho^U_{U_\alpha}(s_2)$ for all α then $s_1 = s_2$.

To do:

- $\mathcal{O}_X$ is a sheaf of rings.
- $\mathcal{F}$ sheaf $\implies$ stalk $\mathcal{F}_p$ at $p \in X$.
- Sheafification $\mathcal{F} \to \mathcal{F}^+$.
- Maps of sheaves.

$X = \text{Spec}A$. Recall $\mathcal{O}_X(X) = A$, $\mathcal{O}_X(U_f) = A_f$ for all $f \in A$. And $\mathcal{O}_X(U) = \varprojlim_{U_f \subset U} \mathcal{O}_X(U_f)$. For $U \subset V$, since $\mathcal{O}_X(V) \subset \prod_{U_f\subset V} \mathcal{O}_X(U_f)$ and $\mathcal{O}_X(U) \subset \prod_{U_f\subset U} \mathcal{O}_X(U_f)$, the restriction map can just be taken to be the projection map from one product to the other.

Definition. *Let $\mathcal{B}$ be a nice basis of the topology of a topological space X. A $\mathcal{B}$-sheaf $\mathcal{F}$ consists of the following data:*

(1) *For every $U \in \mathcal{B}$, a set $\mathcal{F}(U)$.*
(2) *For every inclusion $U \subset V$, $U, V \in \mathcal{B}$, a map $\rho^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ such that*
    (a) *$\mathcal{F}(\varnothing) =$ single element set.*
    (b) *$\rho^U_U = \text{id}$ for all $U \in \mathcal{B}$.*
    (c) *For $U \subset V \subset W$, $U, V, W \in \mathcal{B}$, $\rho^W_U = \rho^V_U \rho^W_V$.*

and the sheaf axiom (on $\mathcal{B}$). If $U = \bigcup U_\alpha$ and $U, U_\alpha \in \mathcal{B}$ for all α and $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $\rho^{U_\alpha}_{U_\alpha\cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(s_\beta)$ for all α, β then there is a unique $s \in \mathcal{F}(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all α.

Proposition (1)**.** $\mathcal{O}_X$ *is a* $\mathcal{B}$*–sheaf where* $\mathcal{B} = \{U_f : f \in A\}$*.*

Proposition (2)**.** *If* $\mathcal{F}_\mathcal{B}$ *is a* $\mathcal{B}$*–sheaf then* $\mathcal{F}$ *extends uniquely to a sheaf* $\mathcal{F}$ *on* X*, and* $\mathcal{F}(U) = \varprojlim_{\mathcal{B} \ni U_\alpha \subset U} \mathcal{F}(U_\alpha)$*.*

Proof. Exercise. ☐

Proof of Proposition (1): (1) Given $U = \bigcup_\alpha U_{g_\alpha}$ where $U, U_{g_\alpha} \in \mathcal{B} = \{U_f : f \in A\}$, WLOG we can take $U = X = \mathrm{Spec} A$. (Exercise: make sure you buy this reduction.)

(2) WLOG the open cover can be taken to be finite. So $X = U_{g_1} \cup \cdots \cup U_{g_r}$, $(g_1, \ldots, g_r) = A$.

Let $s_1, \ldots, s_r$, $s_i \in \mathcal{O}_X(u_{g_i}) = A_{g_i}$, $s_i = t_i/g_i^n$, $t_i \in A$, $1 \leq i \leq r$ and n fixed. *(We are trying to show that the sheaf condition holds so we take arbitrary elements whose restrictions agree. We can take* n *fixed because each* s_i *can be multiplied by an appropriate power of* g_i *as necessary.)* $U_{g_i} \cap U_{g_j} = U_{g_i g_j}$. The sheaf condition says that $t_i/g_i^n = t_j/g_j^n$ in $A_{g_i g_j}$, ie. there exists m such that

$$(g_i g_j)^m (t_i g_j^n - t_j g_i^n) = 0$$

in A. Want to find an $s \in \mathcal{F}(X) = A$ such that $\rho^X_{U_{g_i}}(s) = s_i$ for all i, ie. $s = t_i/g_i^n$ in A_{g_i}.

Let $u_i = t_i g_i^m$. Then $u_i g_j^{m+n} = u_j g_i^{m+n}$ in A. Note that $(g_1^{m+n}, \ldots, g_r^{m+n}) = A$, so there are α_i such that $\sum_{i=1}^r g_i^{m+n} \alpha_i = 1$ for some $\alpha_i \in A$. Let $s = \sum_{i=1}^r u_i \alpha_i \in A$. Then $g_j^{m+n} s = \sum_{i=1}^r g_j^{m+n} u_i \alpha_i = \sum_{i=1}^r u_j g_i^{m+n} \alpha_i = u_j$. So in A_{g_j}, $s = u_j/g_j^{m+n} = t_j/g_j^n$.

So an appropriate s exists. To show uniqueness, need to show that if $\rho^X_{U_{g_j}}(s) = 0$ for all j then $s = 0$. But $\rho^X_{U_{g_j}}(s) = 0$ means $g_j^n s = 0$ for some n. If this is true for all j then $s = 0$ since for each $n \geq 1$, $(g_1^n, \ldots, g_r^n) = A$.

33. Stalks and sheafification

To do list: stalks, sheafification, ringed spaces, Spec$A$, maps of sheaves and ringed spaces, locally ringed spaces, schemes.

33.1. **Stalks.** Let $\mathcal{F}$ be a presheaf on a topological space X and $p \in X$.

Definition. *A* germ *of* $\mathcal{F}$ *at* p *is an equivalence class of pairs* (U, s)*,* $s \in \mathcal{F}(U)$ *with* $p \in U$ *and* U *open. The equivalence relation is* $(U, s) \sim (V, t)$ *if there exists* $W \subset U \cap V$ *such that* $\rho^U_W(s) = \rho^V_W(t)$*.*

Let $\mathcal{F}_p$ be the set of germs of $\mathcal{F}$ at p. If $\mathcal{F}$ is a sheaf of rings, groups, A–modules... then so is $\mathcal{F}_p$. Note: can also be defined as $\varinjlim_{p\in U} \mathcal{F}(U)$ (the "union" limit instead of the "intersection" limit).

Example. Let $X \subset \mathbb{A}^n$ be an irreducible affine variety. Then we have a sheaf of rings $\mathcal{O}_X$, with $\mathcal{O}_X(U) = \{f \in k(X) : f$ is regular at every $p \in U\}$. If $p \in X$, what is $(\mathcal{O}_X)_p$? Answer: $\mathcal{O}_{X,p} = \{f \in k(X) : f$ regular at $p\} = k[X]_{m_p}$.

Example. Let $X = \mathrm{Spec} A$ and $p \in X$. What is $(\mathcal{O}_X)_p$? Answer: $\mathcal{O}_X, p = \varinjlim_{U\ni p} \mathcal{O}_X(U) = \varinjlim_{U_\alpha \ni p,\, U_\alpha \in \mathcal{B}} \mathcal{O}_X(U_\alpha) = \varinjlim_{f\notin p} \mathcal{O}_X(U_f) = \varinjlim_{f\notin p} A_f = A_p$ (check these statements).

33.2. **Sheafification.** Suppose $\mathcal{F}$ is a sheaf on X. Suppose $s \in \mathcal{F}(U)$. Consider the function $p \mapsto s_p = (U, s)$ in $\mathcal{F}_p$. Denote this function $U \to \sqcup_{p\in U} \mathcal{F}_p$ by $\mathrm{fcn}(s)$. Question: if $s, t \in \mathcal{F}(U)$ and $\mathrm{fcn}(s) = \mathrm{fcn}(t)$, is $s = t$?

Answer: yes. Since U has an open cover by sets W_p with $s|_{W_p} = t|_{W_p}$, so by the sheaf axiom $s = t$.

Let

$$\begin{aligned}\mathcal{F}^+(U) := \{&\alpha : U \to \sqcup_{p\in U} \mathcal{F}_p : \alpha(q) \in \mathcal{F}_q \ \forall q \in U \text{ and} \\ &\forall p \in U \ \exists \text{ nhd } p \in W \subset U \text{ and } w \in \mathcal{F}(W) \text{ such that } \alpha(p) = w_p \in \mathcal{F}_p \text{ for all } p \in W\}\end{aligned}$$

If $\mathcal{F}$ is a sheaf, we have basically shown $\mathcal{F}^+(U) \cong \mathcal{F}(U)$ for all open U (natural map is $\mathcal{F}(U) \to \mathcal{F}^+(U)$). Really we have a map of sheaves $\mathcal{F} \to \mathcal{F}^+$ which is an isomorhphism if $\mathcal{F}$ is a sheaf (so every sheaf is a sheaf of functions!) Instead, if $\mathcal{F}$ is only a presheaf, then $\mathcal{F}^+$ is still well-defined and if $U \subset V$, then $\mathcal{F}^+(V) \to \mathcal{F}^+(U)$ is restriction of functions. This is a sheaf on X, and we have a map of presheaves $\mathcal{F} \to \mathcal{F}^+$.

Fix presheaf $\mathcal{F}$. Let $\mathrm{sp}(\mathcal{F}) = \sqcup_{p\in X} \mathcal{F}_p$. Define a topology on $\mathrm{sp}(\mathcal{F})$ as follows: for every $s \in \mathcal{F}(U)$, let $\{s_p : p \in U\} \subset \mathrm{sp}(\mathcal{F})$ be an open set.

Exercise. *Show that*

$$\mathcal{F}^+(U) = \{s : U \to \mathrm{sp}(\mathcal{F}) : s \text{ is continuous and } \pi s = \mathrm{id}_U\}$$

where $\pi : \sqcup_{p\in X} \mathcal{F}_p \to X$ *maps* $a \in \mathcal{F}_p$ *to* p.

Definition. *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *consisting of*

(1) X *is a topological space.*

(2) $\mathcal{O}_X$ *is some sheaf of rings on* X*.*

$\mathcal{O}_X$ *is called the* structure sheaf *of* X*.*

Examples.

(1) *Let* $X \subset \mathbb{R}^2$ *open ball,* $\mathcal{O}_X(U) = \{s : U \to \mathbb{R} \text{ cts}\}$*. Then* $(X, \mathcal{O}_X)$ *is a ringed space.*

(2) *Instead,* $\mathcal{O}_X(U) = \{f : U \to \mathbb{R} : f\ C^\infty\}$*. Then* $(X, \mathcal{O}_X)$ *is a ringed space.*

(3) $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$.

(4) $X \subset \mathbb{A}^n$ *affine;* $(X, \mathcal{O}_X)$.

34. Ringed spaces

Ringed spaces; morphisms of these; $\mathrm{Spec}B \to \mathrm{Spec}A$; locally ringed spaces; schemes.

Ringed space $(X, \mathcal{O}_X)$, $\mathcal{O}_X$ a sheaf of rings on X.

Definition. *A morphism of ringed spaces* $(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *consists of the following data:*

(1) $\varphi : X \to Y$ *a continuous map of spaces.*

(2) $\varphi^\#_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}U)$ *for all open* $U \subset Y$*.*

such that for all $U \subset V \subset Y$ *open, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\varphi^\#_V} & \mathcal{O}_X(\varphi^{-1}V) \\ \downarrow{\scriptstyle \rho^V_U} & & \downarrow{\scriptstyle \rho^{\varphi^{-1}V}_{\varphi^{-1}U}} \\ \mathcal{O}_Y(U) & \xrightarrow{\varphi^\#_U} & \mathcal{O}_X(\varphi^{-1}U) \end{array}$$

Definition. *If* $\varphi : X \to Y$ *is a continuous map of topological spaces and* $\mathcal{F}$ *is a sheaf on* X*, define for* $U \subset Y$*,* $(\varphi_*\mathcal{F})(U) = \mathcal{F}(\varphi^{-1}U)$*. For* $U \subset V \subset Y$*, define* $\rho^V_U = \rho^{\varphi^{-1}V}_{\varphi^{-1}U}$*. Then* $\varphi_*\mathcal{F}$ *is a sheaf on* Y*.*

Definition. *Let $\mathcal{F}, \mathcal{G}$ be sheaves (of groups, rings, ...) on X. A* morphism of sheaves *$\psi : \mathcal{F} \to \mathcal{G}$ (of groups, rings, ...) consists of morphisms*

$$\psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for all open $U \subset X$, such that for $U \subset V \subset X$ open,

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\psi_V} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \rho^V_U} & & \downarrow{\scriptstyle \rho^V_U} \\ \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{G}(U) \end{array}$$

commutes.

Alternate definition: a morphism of ringed spaces $(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of

(1) $\varphi : X \to Y$ continuous.

(2) $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ a morphism of sheaves of rings.

Example. Let $X = \text{Spec}A$, $Y = \text{Spec}B$, ringed spaces. Let's consider morphisms $X \to Y$. For example, if $\lambda : A \to B$ is a morphism of rings, define ${}^a\lambda : \text{Spec}B \to \text{Spec}A$, a morphism of ringed spaces, as follows. ${}^a\lambda = (\varphi, \varphi^\#)$ where $\varphi : \text{Spec}B \to \text{Spec}A$ was already defined. For $U_f \subset \text{Spec}A$, $\varphi^\#_{U_f} : \mathcal{O}_Y(U_f) \to \mathcal{O}_X(\varphi^{-1}U_f)$ is the natural map $A_f \to B_{\lambda(f)}$ induced by λ. Here $\mathcal{O}_X(\varphi^{-1}U_f)$ is identified with $U_{\lambda(f)} \subset \text{Spec}B$.

In general for an open U, define $\varphi^\#_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}U)$ as the limit of the above morphisms, which makes sense since $\mathcal{O}_Y(U) = \varprojlim_{U_f \subset U} \mathcal{O}_Y(U_f)$ and $\mathcal{O}_X(\varphi^{-1}U) = \varprojlim_{U_g \subset \varphi^{-1}(U)} \mathcal{O}_X(U_g)$. **Exercise:** find this natural map and then show that $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is a morphism of sheaves. **Exercise:** In general, define a $\mathcal{B}$–morphism of sheaves and show that it extends uniquely to a morphism of sheaves.

Exercise. (1) $X = \text{Spec}B$, $Y = \text{Spec}A$. Suppose $\varphi : X \to Y$ is induced from $\lambda : A \to B$. Show that $\varphi^\#_p : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ given by $(U, s) \mapsto (\varphi^{-1}U, \varphi^\#_U(s))$ is a local homomorphism, ie. $\varphi^\#_p(m_{\varphi(p)}) \subset m_{X,p}$ for all $p \in X$.

(2) Find a morphism of ringed spaces $\varphi : X \to Y$ such that $\varphi_p^\# : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ is not a local homomorphism.
(3) Show that if $(\varphi, \varphi^\#) : \mathrm{Spec}B \to \mathrm{Spec}A$ is a local homomorphism then there is $\lambda : A \to B$ such that $(\varphi, \varphi^\#) = {}^a\lambda$. Note: if it exists then λ will be given by $\varphi^\#_{\mathrm{Spec}A}$.

Definition. *$(X, \mathcal{O}_X)$ is a* locally ringed space *if*

(1) *X is a topological space.*
(2) *$\mathcal{O}_X$ is a sheaf of rings on X and $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$.*

A morphism of locally ringed spaces $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that $\varphi_p^\# : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism for all $p \in X$.

Remark. If $(X, \mathcal{O}_X)$ is a locally ringed space and $U \subset X$ open then $(U, \mathcal{O}_X|_U)$ is a locally ringed space.

Definition. *A* scheme *is a locally ringed space $(X, \mathcal{O}_X)$ such that for all $p \in X$, there exists an open neighbourhood $U \subset X$ of p such that $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}A$ for some ring A, as locally ringed spaces. [Could also just say as ringed spaces, since an isomorphism automatically maps unique maximal ideal to unique maximal ideal.]*

An open set $U \ni p$ such that $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}A$ is called an affine nbd of p.

Definition. *A morphism of schemes is defined to be a morphism as locally ringed spaces, ie. $(\varphi, \varphi^\#) : X \to Y$, $\varphi : X \to Y$ continuous, $\varphi^\# : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ with $\varphi_p^\# : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ a local homomorphism for all p.*

35. Schemes

Last time: $(X, \mathcal{O}_X)$ is a scheme if

(1) it is a locally ringed space.
(2) every pt. $p \in X$ has an open nbd U such that $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}A$ for some ring A.

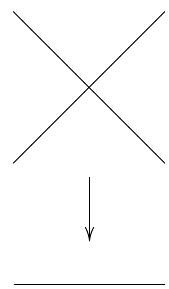
Morphisms of schemes are morphisms of locally ringed spaces.

- Maps to Spec$A$.
- Examples.
- Quasiprojective variety $\rightsquigarrow$ scheme.

- Glueing of schemes and projective space.

Example. $X = \mathrm{Spec} k[x,y]/(x^2-y^2) = \mathrm{Spec}(k[x,y]/(x-y)) \bigcup \mathrm{Spec}([k(x,y]/(x+y))$ is irreducible decomposition of X.

Map $\pi : X \to Y = \mathrm{Spec} k[x]$ corresponds to the natural inclusion $k[x] \hookrightarrow k[x,y]/(x^2-y^2)$. Picture:



Definition. *For $\varphi : \mathrm{Spec} B \to \mathrm{Spec} A$, corresponding to a ring homomorphism $\lambda : A \to B$, if $p \in Y$ then the fibre X_p over p is by definition*

$$X_p = V(\lambda(p)B).$$

In our example, $X_{(x-1)} = V((x-1) \subset k[x,y]/(x^2-y^2)) = \mathrm{Spec}(k[x,y]/(x-1, 1-y^2)) = V(x-1, y-1) \bigcup V(x-1, y+1)$. The fibre $X_0 = X_{(x)}$ is $\mathrm{Spec}(k[x,y]/(x, x^2-y^2)) = \mathrm{Spec}(k[x,y]/(x, y^2))$. This is known as a fat point.

In general, consider

$$k[y]/(y^3) \to k[y]/(y^2) \to k[y]/(y)$$

The associated maps are homeomorphisms of Spec's. They correspond to the inclusions

$$\bullet \hookleftarrow \bullet \hookleftarrow \bullet$$

of a point into a thicker point into a thicker point.

Proposition. *If X is a scheme and A is a ring then morphisms $(\varphi, \varphi^{\#}) : X \to \mathrm{Spec} A$ are in one-to-one correspondence with ring homomorphisms $A \to \mathcal{O}_X(X)$.*

Proof. Exercise. □

If $\varphi : X \to \mathrm{Spec} A$ is a morphism of schemes then $\mathcal{O}_X$ is a sheaf of A–algebras.

Examples. (1) $X \to \mathrm{Spec}\mathbb{Z}$ always exists.

(2) $X \to \mathrm{Spec} k$, k a field. Means that $\mathcal{O}_X(U)$ is a k–algebra so $\mathcal{O}_{X,p}$ is a k–algebra for all $p \in X$. Called a k–scheme or scheme over k.

(3) $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$. In general, $\mathbb{A}^n_A = \mathrm{Spec} A[t_1, \ldots, t_n]$. What is a map $X \to \mathbb{A}^1_k$? Answer: a family of schemes.

Terminology: if $X \to \mathrm{Spec} A$ we call X an A–scheme. The maps in the category of A–schemes are scheme maps $X \to Y$ which commute with the given maps $X, Y \to A$.

35.1. **Quasiprojective varieties.** Let X be an affine variety, $X \subset \mathbb{A}^n$. To give X, you have to give coordinate ring of $X \rightsquigarrow \mathrm{Spec} k[X]$, a scheme. Now suppose instead that X is quasiprojective. So let's define a scheme $\tilde{X}$. As a set, $\tilde{X}$ = set of all irreducible closed subvarieties of X. If $U \subset X$ is open, $\tilde{U}$ = set of all irreducible subvarieties of U. If $Z \subset U$ closed, then $X \supset \overline{U} \supset \overline{Z}$, so $\overline{Z}$ is closed in X. Consider $\tilde{U} = \{\overline{Z} : Z \text{ irred subvariety of } U\}$. Declare this to be open. Put $\mathcal{O}_{\tilde{X}}(\tilde{U}) = k[U]$. Check this gives a sheaf and that $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a k–scheme.

36. Products

Quasiprojective varieties $\rightsquigarrow$ schemes.

X quasiprojective.

$\tilde{X}$ set of all irreducible closed subvarieties.

U' = set of all irreducible closed subvarieties of open U.

$\tilde{U} = \{\overline{Z} \subset X : Z \in U'\}$.

$$\mathcal{O}_{\tilde{X}}(\tilde{U}) = \mathcal{O}_X(U) = k[U]$$

Exercise. *This $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a scheme, and in fact a k–scheme.*

How do regular maps correspond? If $f : X \to Y$, X, Y quasiprojective, then $\tilde{f} : \tilde{X} \to \tilde{Y}$ is defined by putting, for Z an irreducible closed subvariety of X, $f : Z \mapsto \overline{f(Z)} \in \tilde{Y}$.

We also need $\tilde{f}^{\#} : \mathcal{O}_{\tilde{Y}} \to \tilde{f}_* \mathcal{O}_{\tilde{X}}$. For $\tilde{U} \subset \tilde{Y}$, need $\mathcal{O}_{\tilde{Y}}(\tilde{U}) \to \mathcal{O}_{\tilde{X}}(\tilde{f}^{-1}\tilde{U})$. But we have a map $k[U] \to k[f^{1-}U]$ and can use this one.

Exercise. *This gives an isomorphism with the category of quasiprojective varieties over k with its image in the category of k–schemes.*

Next time: Open subschemes, closed subschemes, reduced subschemes, finite type.

Now: Products.

36.1. **Products.**

Example. Let X, Y and S be sets.

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

Define $X \times_S Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$. The diagram commutes.

Some important special cases:

(1) $S = \text{pt}$. Then $X \times_S Y = X \times Y$.

(2) $X, Y \subset S$ and α, β the inclusions. Then $X \times_S Y = X \cap Y$.

(3) $Y \subset S$, β inclusion. Then $X \times_S Y = \alpha^{-1}(Y)$. $Y = p \in S$. Then $X \times_S \{p\} = \alpha^{-1}(p)$, the fibre.

(4) $X = Y$. Then $X \times_S X = \{(x, y) : \alpha(x) = \beta(y)\}$.

Example: $X = \text{Spec} k[t]$, $Y = \text{Spec} k[s]$. Want $X \times Y$ to be $\text{Spec}(k[s, t]) = \mathbb{A}^2$. This is too big to just have $X \times Y$ as a point set.

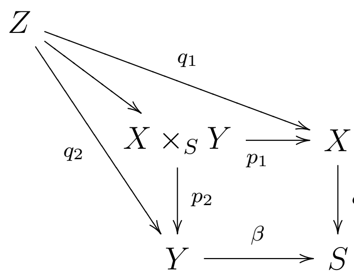
Universal property: For sets,

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

$X \times_S Y$ is the unique up to iso. set such that whenever the diagram

$$\begin{array}{ccc} Z & \xrightarrow{q_1} & X \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

commutes, there exists a unique map $Z \to X \times_S Y$ such that



$(*)$

commutes.

Definition. *Let X, Y, S be schemes, X, Y are S–schemes and $\alpha : X \to S$, $\beta : Y \to S$. A* fibre product *$X \times_S Y$ is a scheme together with maps of schemes $p_1 : X \times_S Y \to X$, $p_2 : X \times_S Y \to Y$, such that*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

commutes, and for all S–schemes $Z \to S$ and maps $q_1 : Z \to X$ and $q_2 : Z \to Y$ such that

$$\begin{array}{ccc} Z & \xrightarrow{q_1} & X \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

commutes, there exists a unique morphism $Z \to X \times_S Y$ such that $(*)$ commutes.

Proposition. *If $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$ and $S = \mathrm{Spec}R$, then $X \times_S Y$ exists and is* $\mathrm{Spec}(A \otimes_R B)$.

Proof. (Sketch). Consider the maps of sheaves on global sections. Get:

$$\begin{array}{ccc} ?? & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & R \end{array}$$

Where ?? is the pushforward in category of rings. Turns out to be $A \otimes_R B$, since the universal property for $X \times_S Y$ translates to the universal mapping property for $A \otimes_R B$. ☐

Theorem. *If*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\beta} & S \end{array}$$

are maps of schemes then the fibre product $X \times_S Y$ *exists.*

Examples. (1) $\mathbb{A}^1_k \times_{\mathrm{Spec}k} \mathbb{A}^1_k = \mathrm{Spec}(k[s] \otimes_k k[t]) = \mathrm{Spec}(k[s,t]) = \mathbb{A}^2_k$.

Exercise: What is $\mathbb{A}^1_k \times_{\mathrm{Spec}\mathbb{Z}} \mathbb{A}^1_k$?

(2) $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}(A) = \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n] \otimes_{\mathbb{Z}} A) = \mathbb{A}^n_A$.

36.2. **Fibres.** If Y is a scheme, let $k(p) = \mathcal{O}_{Y,p}/m_{Y,p}$ for $p \in Y$. The inclusion $p \hookrightarrow Y$ may be regarded as $\mathrm{Spec}(k(p)) \to Y$. Then there is a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k(p)) & \longrightarrow & Y \end{array}$$

Definition. *The fibre* $f^{-1}(p) := X \times_Y \mathrm{Spec}(k(p))$.

37. Closed subschemes, reduced subschemes, separated schemes

If $X \subset Y$ are affine varieties then $k[Y] \twoheadrightarrow k[X]$ so $k[X] = k[Y]/I$ for some ideal $I \subset Y$.

Definition. *X, Y schemes. Then $\varphi : X \to Y$ is called a closed embedding if every point $p \in Y$ has an affine neighbourhood U such that $\varphi^{-1}(U) \subset X$ is affine and $\varphi^{\#} : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is surjective.*

Main example: if $B = A/I$ then $\varphi : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a closed embedding.

Proposition. *If $\varphi : X \to \mathrm{Spec}A$ is a closed embedding then X is affine and $X \cong \mathrm{Spec}(A/I)$ for some ideal $I \subset A$.*

37.1. **Reduced subschemes.**

Example. Consider $k[x,y]/(x^2) \to k[x,y]/(x) \to 0$. So $\mathrm{Spec}(k[x,y]/(x)) \to \mathrm{Spec}(k[x,y]/(x^2))$ is a closed embedding. Regard as the embedding of a line in a double line.

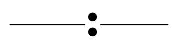
If $X = \text{Spec}A$, define $X_{red} = \text{Spec}(A/\sqrt{(0)})$. Then $X_{red} \subset X$ is a closed embedding.

Proposition. *If X is any scheme then we can define a reduced scheme X_{red}. (If $X = \bigcup_\alpha \text{Spec}(A_\alpha)$ then glue together $\text{Spec}(A_\alpha/\sqrt{(0)})$ to get X_{red}.)*

37.2. **Separated schemes.** A topological space X is Hausdorff if for all $p, q \in X$ with $p \neq q$, there exist open sets $U \ni p$ and $V \ni q$ with $U \cap V = \varnothing$.

Proposition. *X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed in the product topology.*

Bad example: The bug-eyed line.

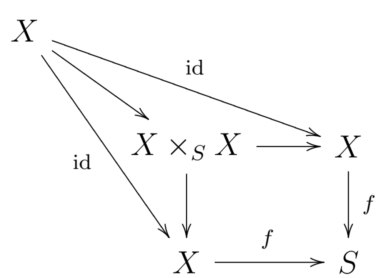
$X =$ 

X is covered by two open affine subsets $X_1 \cong \mathbb{A}^1_k$, $X_2 \cong \mathbb{A}^1_k$.

Glue X_1 (origin o_1) and X_2 (origin o_2) along $X_1 \setminus o_1 \cong X_2 \setminus o_2$ via the map $t \mapsto t$ [$t \mapsto t^{-1}$ gives $\mathbb{P}^1_k$].

Why is this a bad space? Consider the inclusions $\phi_i : X_i \hookrightarrow X$, $i = 1, 2$. Then $\{x \in \mathbb{A}^1 : \phi_1(x) = \phi_2(x)\} = \mathbb{A}^1 \setminus 0$ is not closed.

In general, given $f : X \to S$, there exists a unique $\Delta : X \to X \times_S X$ such that $p_1\Delta = p_2\Delta = \text{id}_X$.



Definition. *$f : X \to S$ is* separated *if $\Delta : X \to X \times_S X$ is a closed embedding. We also say that X is S–separated, and if $S = \text{Spec}\mathbb{Z}$ we say that X is separated.*

Example (continued): $\Delta : X \to X \times_k X := X \times_{\mathrm{Spec}(k)} X$. This is covered by the following affine sets $\cong \mathbb{A}^2$ with the given origins.

$$\begin{array}{ll} X_1 \times X_1 & (o_1, o_1) \\ X_1 \times X_2 & (o_1, o_2) \\ X_2 \times X_1 & (o_2, o_1) \\ X_2 \times X_2 & (o_2, o_2) \end{array}$$

$(X_1 \times X_2) \cap \Delta(X) = \{(x, y) \in \mathbb{A}^2 : x \in X_1 \setminus o_1 \cong X_2 \setminus o_2\}$ not a closed subspace. So X is not separated.

$\overline{\Delta(X) \cap (X_1 \times X_2)} = \Delta(X) \cap (X_1 \times X_2) \bigcup \{(o_1, o_2)\}$.

$\overline{\Delta(X)}$ = affine line with four origins. $\Delta(X)$ = affine line with two origins.

$\mathcal{O}_{X,o_1} = \mathcal{O}_{X,o_2} \subset k(t)$. No function can tell o_1 and o_2 apart.

Unravel definition in case $f : X \to Y$, $X = \mathrm{Spec}B$, $Y = \mathrm{Spec}A$. f corresponds to ring map $\lambda : A \to B$. We get $\Delta^{\#} : B \otimes_A B \to B$, which maps $b_1 \otimes b_2$ to $\Delta^{\#}(b_1 \otimes 1)\Delta^{\#}(1 \otimes b_2) = b_1 b_2$. This is a surjective map and $\ker \Delta^{\#} = \langle b \otimes 1 - 1 \otimes b : b \in B \rangle$. Therefore the map $\Delta : X \to X \times_Y X$ is a closed embedding. So f is separated.

Basic facts:

Proposition. *Suppose X is a scheme over* $\mathrm{Spec}B$ *(X is a B–scheme). If $X = \bigcup U_\alpha$ is an open affine cover such that*

(1) *$U_\alpha \cap U_\beta$ is affine for all α, β.*
(2) *$\mathcal{O}_X(U_\alpha \cap U_\beta)$ is generated by the image of $\mathcal{O}_X(U_\alpha) \to \mathcal{O}_X(U_\alpha \cap U_\beta)$ and $\mathcal{O}_X(U_\beta) \to \mathcal{O}_X(U_\alpha \cap U_\beta)$.*

The X is separated over B.

Notes:

- The converse holds too. [If X separated then any open cover U_α satisfies the two properties.]
- doesn't depend on B.

$\mathcal{O}_X$–modules and coherent sheaves and their cohomology.

Definition. *$(X, \mathcal{O}_X)$ a ringed space. A sheaf $\mathcal{F}$ is called an $\mathcal{O}_X$–module (or a sheaf of $\mathcal{O}_X$–modules) if*

(1) *$\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$–module for all open $U \subset X$.*
(2) *For all open $U \subset V \subset X$, $\rho = \rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ is compatible with the module structure. That is, if $a \in \mathcal{O}_X(V)$ and $s \in \mathcal{F}(V)$ then $(as)|_U = a|_U \cdot s|_U$.*

Examples. (1) $\mathcal{O}_X$, the free module $\mathcal{O}_X^r$.
(2) If X is an irreducible variety with function field $K(X)$, then $\mathcal{K}(X)(U) := K(X)$ for all U is the constant sheaf of rational functions.
(3) $X = \mathrm{Spec}A$. Let M be an A–module. Define an $\mathcal{O}_X$–module $\widetilde{M}$ by (for $f \in A$) $\widetilde{M}(U_f) = M_f := M \otimes_A A_f$, an $\mathcal{O}_X(U_f) = A_f$–module.

Need to check: this defines a $\mathcal{B}$–sheaf and that $\widetilde{M}$ is an $\mathcal{O}_X$–module. If $X \subset \mathbb{A}^n$, can also define $\widetilde{M}$ for a $k[X]$–module M in the same way.

Glueing sheaves on a topological space [on a scheme]: given X a topological space [scheme] and U_i an open cover [open affine cover] and $\mathcal{F}_i$ a sheaf on U_i [$\mathcal{O}_{U_i} = \mathcal{O}_X|_{U_i}$–module] for all i, and isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ [isomorphisms of $\mathcal{O}_X|_{U_i \cap U_j}$–modules], such that

(1) $\varphi_{ii} = 1$.
(2) $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

then there exists a sheaf $\mathcal{F}$ on X [an $\mathcal{O}_x$–module] such that $\mathcal{F}|_{U_i} = \mathcal{F}_i$ for all i.

Definition. *Let X be a scheme. An $\mathcal{O}_X$–module $\mathcal{F}$ is called* coherent *if there exists an open cover $\{U_i\}$, $U_i = \mathrm{Spec}A_i$ of X and finitely-generated A_i–modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ on U_i.*

Fact.

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module and if $U \subset X$ is affine and open, $U = \mathrm{Spec}A$ with A Noetherian, then $\mathcal{F}|_U = \widetilde{M}$ for some finitely-generated A–module M.

Example. $X = \mathbb{P}^n_k$, $S = k[x_0, \ldots, x_n]$. X covered by open affine schemes U_i, $0 \le i \le n$.

$$U_i := \operatorname{Spec}\left(k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]\right).$$

These glue (read glueing of schemes section in Shafarevich vol II). Can also define $\mathbb{P}^n_A$, $U_i = \operatorname{Spec} A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$.

$$\mathcal{O}_X(U_i) = k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] = \left(S\left[\frac{1}{x_i}\right]\right)_0$$

(the zeroth graded component of graded ring). For $f \in S$ homogeneous, $\{U_f\}$ is a basis of the topology and we define $\mathcal{O}_X(U_f) = S[1/f]_0$.

Let M be a graded S–module, eg. $M = I \subset S$, I homogeneous. Or eg. $M = S/I$ for such an I. an example is the module $M = S(d)$ such that M is the module S but shifted, $S(d)_e = S_{d+e}$. For example $S(-1)$ is generated by one element, 1, of degree one, since $1 \in S_0 = S(-1)_1$. And $S(-1)_0 = S_{-1} = 0$.

Want to define a coherent $\mathcal{O}_X$–module $\widetilde{M}$ for $X = \mathbb{P}^n$. We put $\widetilde{M}(U_f) = M\left[\frac{1}{f}\right]_0 = \left(M \otimes_S S\left[\frac{1}{f}\right]\right)_0$. Since M was finitely-generated, this is a f.g. $\mathcal{O}_X(U_f)$–module.

Check:

$\widetilde{M}$ is a $\mathcal{B}$–sheaf, $\mathcal{B} = \{U_f : f \text{ homogeneous}\}$, on $\mathbb{P}^n$.

$\widetilde{M}$ extends uniquely to a coherent $\mathcal{O}_X$–module.

Fact.

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module on $X = \mathbb{P}^n$ then there exists a f.g. graded S–module M such that $\widetilde{M} = \mathcal{F}$. Note that M is not unique!

Important special case: $M = S(d)$. Then let $\mathcal{O}_X(d) := \widetilde{S(d)}$.

Example. Let X be an irreducible variety, ie. the scheme associated to such. So $k(X)$ exists. Let $D = \{(U_i, f_i)\}_{i=1}^r$ be a Cartier divisor on X, $f_i \in k(X)^*$, U_i affine. Define a coherent $\mathcal{O}_X$–module $\mathcal{O}_X(D)$.

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i}\mathcal{O}_X|_{U_i}$$

for $1 \le i \le r$ and glue using the obvious maps. Check:

(1) $\mathcal{O}_X(D)$ is a coherent $\mathcal{O}_X$–module and in fact is locally free of rank 1.
(2) Compute $\mathcal{O}_X(D)(X) =: H^0(X, \mathcal{O}_X(D))$. (This is a subset of $k(X)$.)
(3) $\mathcal{O}_X(-D)$, $\mathcal{O}_X(-D)|_{U_i} = f_i\mathcal{O}_X|_{U_i}$.

39. Sheaf cohomology

Important special case. $X \subset \mathbb{P}^n$ projective irreducible variety. $I \subset S = k[x_0, \ldots, x_n]$ homogeneous ideal. For $f \in S$ homogeneous, $\mathcal{O}_X(U_f) = \left(\frac{S}{I}\left[\frac{1}{f}\right]\right)_0$. $\mathcal{O}_X$ is the sheaf of regular functions on X.

If M is a graded S/I–module, then $\widetilde{M}$ is an $\mathcal{O}_X$–module, defined on basic open sets by $\widetilde{M}(U_f) = (M \otimes_S S[1/f])_0$. If M is f.g then $\widetilde{M}$ is a coherent $\mathcal{O}_X$–module.

Examples. (1) $\widetilde{S/I} = \mathcal{O}_X$ (where S/I is thought of as an S/I–module, not an S–module. Here $\widetilde{M}$ is ambiguous, but if you thought of it as an S–module then you'd get the sheaf $i_*\mathcal{O}_X$ on $\mathbb{P}^n$.)

(2) $\widetilde{I} = \mathcal{I}$, ideal sheaf defining X (a sheaf on $\mathbb{P}^n$).

39.1. Maps of $\mathcal{O}_X$–modules.

Definition. *A map (or morphism) $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_X$–modules if:*

(1) *$\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$–modules.*
(2) *$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a map of $\mathcal{O}_X(U)$–modules for all open $U \subset X$.*

Enough to give φ_U for $U \in \mathcal{B}$, a base of the topology.

Given $\varphi : \mathcal{F} \to \mathcal{G}$ a map of $\mathcal{O}_X$–modules, define the *kernel* by

$$(\ker \varphi)(U) = \ker(\varphi_U)$$

Check: $\ker \varphi$ is a sheaf and an $\mathcal{O}_X$–module.

The *image* is more subtle. Define $\mathcal{H}'(U) := \text{image}(\varphi_U)$.

Exercise. *$\mathcal{H}'$ is a presheaf. Find an example of a φ such that $\mathcal{H}'$ is not a sheaf.*

Let $\text{im}(\varphi) =$ sheafification of $\mathcal{H}'$. Then $\text{im}(\varphi)$ is an $\mathcal{O}_X$–module. Similarly, we may define the cokernel of φ

$$\text{coker}(\varphi) = \text{ sheafification of the presheaf } U \mapsto \mathcal{G}(U)/\text{im}(\varphi_U).$$

Key fact:

$$\mathcal{F} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\alpha} \mathcal{H}$$

is an exact sequence of $\mathcal{O}_X$–modules iff

$$\mathcal{F}_p \xrightarrow{\beta_p} \mathcal{G}_p \xrightarrow{\alpha_p} \mathcal{H}_p$$

is exact as $\mathcal{O}_{X,p}$–modules for all $p \in X$.

Other key fact: If

$$0 \longrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \longrightarrow 0$$

is a s.e.s of $\mathcal{O}_X$–modules, then for all open $U \subset X$,

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\beta} \mathcal{G}(U) \xrightarrow{\alpha} \mathcal{H}(U)$$

is exact as $\mathcal{O}_X(U)$–modules.

One way to define cohomology of sheaves is to consider $\mathcal{F} \mapsto \mathcal{F}(X)$, the global sections functor. This is left exact, so we can take the derived functor to get $H^i(X, \mathcal{F})$ for $i \geq 0$. For $i = 0$, $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Assume $X \subset \mathbb{P}^n$ is an irreducible projective variety and $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module. Then $H^i(X, \mathcal{F})$ (also denoted $H^i(\mathcal{F})$) satisfies

(1) $H^i(X, \mathcal{F})$ is a vector space over k, and is finite-dimensional. Its dimension is denoted $h^i(X, \mathcal{F})$ or $h^i(\mathcal{F})$.

(2) $H^i(X, \mathcal{F}) = 0$ for all $i > \dim(X)$ and for all $i < 0$ (also equals zero for all $i > \dim(\mathrm{supp}(\mathcal{F}))$.

(3) Long exact sequence: if

$$0 \longrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of coherent $\mathcal{O}_X$–modules then

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to H^1(\mathcal{G}) \to \cdots \to H^{\dim(X)}(\mathcal{H}) \to 0$$

is exact.

(4) Let $X = \mathbb{P}^n$, $\mathcal{O}_X(d) := \widetilde{S(d)}$, $S = k[x_0, \ldots, x_n]$. Then

(a) $H^0(\mathcal{O}_X(d)) \cong S_d$ (homogeneous forms of degree d), and therefore is 0 for $d < 0$.

(b) $H^i(\mathcal{O}_X(d)) = 0$, $0 < i < n$ for all d.

(c) $H^n(\mathcal{O}_X(-n-1-d)) = S_d^*$ (the k–vector space dual of S_d). In particular, $H^n(\mathcal{O}_X(d)) = 0$ for $d \geq -n$.

Need exact sequences with degree 0 maps. Note:

$$\widetilde{(\cdot)} : (\text{graded } S/I - \text{modules}) \to \text{ coherent } \mathcal{O}_X - \text{modules}$$

(for $X \subset \mathbb{P}^n$, $I \subset S$ homogeneous ideal) is a functor. It is also exact, because localisation is.

One such exact sequence:

$$0 \to I \to S \to S/I \to 0$$

gives

$$0 \to \widetilde{I} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

where technically $\mathcal{O}_X$ means $i_*\mathcal{O}_X$ as an $\mathcal{O}_{\mathbb{P}^n}$–module.

Example. $X = \mathbb{P}^2$, $F \in k[x,y,z]$ homogeneous of degree 3. $C = V(F) \subset \mathbb{P}^2$, elliptic curve.

$$0 \to (F) \to S \to S/(F) \to 0.$$

Here, $(F) \cong S(-3)$ because generator has degree 3. So

$$0 \to S(-3) \xrightarrow{\cdot F} S \to S/(F) \to 0.$$

This gives

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

(where $\mathcal{O}_C$ really means $i_*\mathcal{O}_C$, a sheaf on $\mathbb{P}^2$).

Exercise: find $h^i(\mathcal{O}_C)$. For next time, do it if $\deg(F) = d$ in general.

40. Serre duality

$C \subset \mathbb{P}^2$ a curve $V(F)$, $\deg(F) = d$. Calculate $h^0(\mathcal{O}_C)$, $h^1(\mathcal{O}_C)$. $S = k[x,y,z]$, $I = (F)$.

$$0 \to S(-d) \xrightarrow{\cdot F} S \to S/(F) \to 0$$

gives

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

or really the last term is $i_*\mathcal{O}_C$ where $i : C \hookrightarrow \mathbb{P}^2$. We get

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^2}(-d)) \to H^0(\mathcal{O}_{\mathbb{P}^2}) \to H^0(\mathcal{O}_C) \to H^1(\mathcal{O}_{\mathbb{P}^2}(-d)) \to \cdots$$

We have from last time: $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = S_d$; $H^i(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$, $0 < i < n$, all d; $H^n(\mathcal{O}_{\mathbb{P}^n}(d)) = S^*_{-n-d-1}$. So $H^0(\mathcal{O}_{\mathbb{P}^2}(-d)) = 0$ and $H^1(\mathcal{O}_{\mathbb{P}^2}(-d)) = 0$, so $H^0(\mathcal{O}_C) = k$. The next part of the long exact sequence gives

$$0 \to H^1(\mathcal{O}_C) \to H^2(\mathcal{O}_{\mathbb{P}^2}(-d)) \to H^2(\mathcal{O}_{\mathbb{P}^2}) \to H^2(\mathcal{O}_C) \to 0$$

(since $3 > \dim(\mathbb{P}^2)$). Using $H^2(\mathcal{O}_{\mathbb{P}^2}) = 0$, we get $H^2(\mathcal{O}_C) = 0$ and $H^1(\mathcal{O}_C) \cong H^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = S^*_{-n-1+d} = S^*_{d-3}$. Therefore

$$h^1(\mathcal{O}_C) = \dim(S^*_{d-3}) = \binom{2+(d-3)}{2} = \binom{d-1}{2}.$$

Definition. *If $\mathcal{F}$ is coherent on X, X projective, then the* Euler characteristic

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{F}).$$

Definition. *The* arithmetic genus *of $X \subset \mathbb{P}^n$ is*

$$p_a(X) := (-1)^{\dim(X)}(\chi(X) - 1).$$

If X is connected and $\dim(X) = 1$ then $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X)$. So $p_a(X) = (-1)(1 - h^1(\mathcal{O}_X) - 1) = h^1(\mathcal{O}_X)$.

$X \subset \mathbb{P}^n$ irreducible, smooth, projective. $D = \{(U_i, f_i)\}$ Cartier divisor on X. We defined $\mathcal{O}_X(D)$ with $\mathcal{O}_X(D)(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i) \subset k(X)$. Global sections:

$s \in H^0(\mathcal{O}_X(D)) = \mathcal{O}_X(D)(X)$. Let $s_i = s|_{U_i}$. Let $s_i = g_i/f_i$ with $g_i \in k(X)$ regular on U_i. Note that in $k(X)$, $s_1 = \cdots = s_r = s \in k(X)$.

$$H^0(\mathcal{O}_X(D)) \hookrightarrow k(X)$$

$$s \mapsto s_i$$

for any i. What is the image of this inclusion?

$\operatorname{div}(s)|_{U_i} + D = \operatorname{div}(s_i) + D$ on U_i. This equals $\operatorname{div}(g_i)|_{U_i} - \operatorname{div}(f_i)|_{U_i} + D$. But $D = \operatorname{div}(f_i)|_{U_i}$ on U_i. So since g_i is regular on U_i we get $\operatorname{div}(s)|_{U_i} + D \geq 0$.

Therefore, $H^0(\mathcal{O}_X(D)) \cong L(D)$. So $h^0(\mathcal{O}_X(D)) = \ell(D)$.

40.1. **Serre duality.** Reference: Serre (FAC) 1955.

If $X \subset \mathbb{P}^N$ smooth, irred, projective, $\dim(X) = n$. Then

$$H^{n-i}(\mathcal{O}_X(D)) \cong H^i(\mathcal{O}_X(K_X - D))^*$$

for all i.

Example. (1) X = smooth projective curve in $\mathbb{P}^n$. $k = H^0(\mathcal{O}_X) = H^1(K_X)^*$, (taking $D = 0$ in the statement of Serre duality). [Notation: $h^i(D) := h^i(\mathcal{O}_X(D))$.] So $h^1(K_X) = 1$ and $h^0(\mathcal{O}_X) = 1$.

(2) $H^1(\mathcal{O}_X) = H^0(\mathcal{O}_X(K_X))^* = L(K_X)^*$. So $h^1(\mathcal{O}_X) = g$, the genus of X $(= \ell(K_X))$.

Theorem. *For X a smooth projective irreducible curve of genus g,*

$$p_a(X) = g = h^0(K_X) = h^1(\mathcal{O}_X).$$

Theorem (Restatement of RR for curves)**.** *For X a smooth irreducible projective curve of genus g, recall that the Riemann-Roch Theorem says $\ell(D) - \ell(K - D) = \deg(D) - g + 1$. So $h^0(D) - h^0(K - D) = h^0(D) - h^1(D) = \chi(\mathcal{O}_X(D))$ and*

$$\chi(\mathcal{O}_X(D)) = \deg(D) + \chi(\mathcal{O}_X).$$

Theorem (Serre)**.** *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module for X projective. Then*

(1) *$H^i(X, \mathcal{F})$ is finite-dimensional over k.*
(2) *$H^i(X, \mathcal{F}(d)) = 0$ for all $i > 0$, for $d >> 0$.*

40.2. **Hilbert polynomials.** $\mathcal{F}$ coherent on X projective $\subset \mathbb{P}^n$.

Theorem. *$m \mapsto \chi(\mathcal{F}(m))$ is a polynomial in m, denoted $p_{\mathcal{F}}(m)$.*

For m large enough, $p_{\mathcal{F}}(m) = h^0(\mathcal{F}(m))$.

If $\mathcal{F} = \mathcal{O}_X$ then $m \mapsto h^0(\mathcal{O}_X(m))$ is polynomial for $m >> 0$. Let $p_X(m) := \chi_X(m)$. This is a polynomial. For $m >> 0$, $p_X(m) = h^0(\mathcal{O}_X(m))$.