MATH 413 FINAL EXAM

Math 413 final exam, 13 May 2008. The exam starts at 9:00 am and you have 150 minutes. No textbooks or calculators may be used during the exam. **This exam is printed on both sides of the paper.** *Good luck!*

(1) **(20 marks)** Let $X = (0, 1] \subset \mathbb{R}$. State whether each of the following statements about X is true or false, giving a brief reason for each answer.

(a) X is bounded. *True. For all $x \in X$, $|x| \leq 1$.*

(b) X can be written as a countable union of open sets. *False. Any union of open sets is open, but X is not open.*

(c) X is compact. *False. X is not closed (it does not contain the cluster point 0), so by the Heine-Borel Theorem cannot be compact.*

(d) There is a point $x_0 \in X$ at which the function $f(x) = \log(x) + x^5 - 8x^4 - 3$ achieves its supremum on X (that is, $f(x_0) = \sup\{f(x) : x \in X\}$). *True. Since $\log(x) \to -\infty$ as $x \to 0$, we have also that $f(x) \to -\infty$ as $x \to 0$. So there exists $n \in \mathbb{N}$ such that $f(x) < f(1)$ if $x < \frac{1}{n}$. Therefore, $\sup_{x\in X} f(x) = \sup_{x\in[\frac{1}{n},1]} f(x)$ and this is attained by f since $[\frac{1}{n}, 1]$ is a compact set.*

(2) **(20 marks)** Let $A \subset \mathbb{R}$. Recall that a function $f : A \to \mathbb{R}$ is said to satisfy a Lipschitz condition on A if there is some $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Let $n \in \mathbb{N}$. Show that the function $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = \sqrt{x + \frac{1}{n}}$ satisfies a Lipschitz condition on $[0, 1]$.

(Hint: you may wish to use the fact that for all $a, b > 0$, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.)

We need to show that there exists M such that

$$|f_n(x) - f_n(y)| \leq M|x - y|$$

for all $x, y \in [0,1]$. *Using the hint, if* $x \geq y$, *we have* $|f_n(x) - f_n(y)| = |\sqrt{x + \frac{1}{n}} - \sqrt{y + \frac{1}{n}}| = \frac{(x+\frac{1}{n})-(y+\frac{1}{n})}{\sqrt{x+\frac{1}{n}}+\sqrt{y+\frac{1}{n}}}$. *Since* $x, y \geq 0$, *we get* $|f_n(x) - f_n(y)| \leq (x-y)/(2/\sqrt{n}) = M|x-y|$ *where* $M = \sqrt{n}/2$.

(b) Show that the sequence of functions $\{f_n\}$ converges uniformly on $[0,1]$ to the function $f(x) = \sqrt{x}$.

We need to show that for all $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that if* $n > N$ *then* $|f_n(x) - f(x)| < \varepsilon$ *for all* $x \in [0,1]$. *We have* $|f_n(x) - f(x)| = \sqrt{x + \frac{1}{n}} - \sqrt{x} = \frac{x+\frac{1}{n}-x}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} \leq \frac{1}{n}/\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}}$. *So given* $\varepsilon > 0$, *we choose* N *with* $1/\sqrt{N} < \varepsilon$, *and then if* $n > N$ *and* $x \in [0,1]$, *we have* $|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \varepsilon$ *as required.*

(c) Show that $f(x) = \sqrt{x}$ does *not* satisfy a Lipschitz condition on $[0,1]$.

There are several ways of doing this. One way is to observe that if $\sqrt{x}$ *was Lipschitz, then* $\frac{\sqrt{x}-0}{x-0} = \frac{1}{\sqrt{x}}$ *would be bounded for* $x \in (0,1)$, *but this is false.*

(d) Now suppose $A \subset \mathbb{R}$ and $f_n : A \to \mathbb{R}$ are functions such that there exists $M \in \mathbb{R}$ such that $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $n \in \mathbb{N}$ and all $x, y \in A$. Suppose the sequence of functions $\{f_n\}$ converges uniformly on A to a function $f : A \to \mathbb{R}$. Show that f satisfies a Lipschitz condition on A. Why does this not contradict your answer to part (c)?

To prove the statement, let $x, y \in A$. *Let* $\varepsilon > 0$. *Choose* $n \in \mathbb{N}$ *such that* $|f_n(z) - f(z)| < \varepsilon$ *for all* $z \in A$. *Then*

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

by the triangle inequality. But by the choice of n, *the first and third terms are* $< \varepsilon$, *while* $|f_n(x) - f_n(y)| \leq M|x-y|$ *by hypothesis. So* $|f(x) - f(y)| \leq 2\varepsilon + M|x-y|$. *But since this holds for all* $\varepsilon > 0$, *we must have* $|f(x) - f(y)| \leq M|x - y|$ *as required. This does not contradict part (c) because although the* f_n *were Lipschitz, they do not have a common bound* M, *so their uniform limit does not necessarily have to be Lipschitz. In fact, parts (a)-(c) show that a uniform limit of Lipschitz functions need not be Lipschitz.*

[TURN OVER]

(3) **(20 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) State what it means for f to be uniformly continuous on $\mathbb{R}$.

f is uniformly continuous on $\mathbb{R}$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

(b) State the Mean Value Theorem.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(x_0)$.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that the derivative f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose there exists $M > 0$ with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Then if $a < b$, then $\frac{f(b)-f(a)}{b-a} = f'(x_0) \leq M$ for some $x_0 \in (a, b)$. So $|f(b) - f(a)| \leq M|b - a|$. Therefore, given $\varepsilon > 0$, if $\delta < \varepsilon/M$ then $|b - a| < \delta$ implies $|f(b) - f(a)| < \varepsilon$. So f is uniformly continuous.

(d) Show that $f(x) = e^{-x^2}$ is uniformly continuous on $\mathbb{R}$.

In view of the previous problem, it suffices to show that the derivative of f is bounded. We have $f'(x) = -2xe^{-x^2}$ for all $x \in \mathbb{R}$. Therefore, $|f'(x)| = 2|x|e^{-x^2}$. Now, there are many ways of showing this is bounded, for example $e^{x^2} \geq 1 + x^2$ for all x, by the power series expansion of e^{x^2}. So it suffices to show that $\frac{2|x|}{1+x^2}$ is bounded. If $|x| \geq 1$ then $\frac{2|x|}{1+x^2} \leq \frac{2}{|x|} \leq 2$ while if $|x| \leq 1$ then also $\frac{2|x|}{1+x^2} \leq 2|x| \leq 2$.

(4) **(20 marks)** Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(a) State what it means for f to be Riemann integrable.

Define a partition P of $[a, b]$ to be a sequence of points $x_0 < x_1 < \ldots < x_n$ with $x_0 = a$ and $x_n = b$. For a partition P, define $|P| = \max(x_i - x_{i-1})$. Define a Cauchy sum of f for the partition P to be a sum of the form $\sum_i f(q_i)(x_i - x_{i-1})$ where $q_i \in [x_{i-1}, x_i]$ for each i.

Then $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there exists a number L such that for all $\varepsilon > 0$ there exists $\delta > 0$, such that whenever P is a partition with $|P| < \delta$, we have $|S(f, P) - L| < \varepsilon$, where $S(f, P)$ is any Cauchy sum of f for the partition P.

(b) Show that if $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then so is $f + g$.

This is difficult to do from the above definition, so instead we use a theorem from the textbook, which says that a function f is Riemann integrable if and only if there exists a sequence of partitions P_j such that $\mathrm{Osc}(f, P_j) \to 0$ *as* $j \to \infty$, *where* $\mathrm{Osc}(f, P_j) = \sum_i (\sup_{q\in[x_{i-1},x_i]} f(q) - \inf_{q\in[x_{i-1},x_i]} f(q))(x_i - x_{i-1})$. *We have* $\sup_{q\in[x_{i-1},x_i]}(f(q) + g(q)) \leq \sup_{q\in[x_{i-1},x_i]} f(q) + \sup_{q\in[x_{i-1},x_i]} g(q)$ *and* $\inf_{q\in[x_{i-1},x_i]}(f(q) + g(q)) \geq \inf_{q\in[x_{i-1},x_i]} f(q) + \inf_{q\in[x_{i-1},x_i]} g(q)$. *From these two facts it follows that for any partition* P, $\mathrm{Osc}(f+g, P) \leq \mathrm{Osc}(f, P) + \mathrm{Osc}(g, P)$ *and so if* $\mathrm{Osc}(f, P_j)$ *and* $\mathrm{Osc}(g, P_j)$ *tend to* 0 *as* $j \to \infty$, *so does* $\mathrm{Osc}(f+g, P_j)$ *(we have left out some details). This theorem can also be proved by various other similar methods.*

(c) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is *not* Riemann integrable.

This time, the best characterization of Riemann integrability to use is the statement that f is Riemann integrable if and only if $\inf_P S^+(f, P) = \sup_P S^-(f, P)$ *where $S^+(f, P)$ and $S^-(f, P)$ denote the upper and lower Riemann sums respectively. For* every *partition* $P = \{x_0 < x_1 < \ldots < x_n\}$ *of* $[0, 1]$ *and each* i, $[x_{i-1}, x_i]$ *contains a point of* $\mathbb{Q}$ *and a point of* $\mathbb{R} \setminus \mathbb{Q}$. *So* $S^+(f, P) = 1$ *while* $S^-(f, P) = 0$. *Therefore,* $\inf_P S^+(f, P) = 0$ *and* $\sup_P S^-(f, P) = 1$, *so f is not Riemann integrable.*

(d) Now let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. Define $F(x) = \int_0^1 f(x+t)dt$. Show that F is continuous on $\mathbb{R}$.

We use the definition of continuity. Let $x_0 \in \mathbb{R}$. *To show that F is continuous at* x_0, *let* $\varepsilon > 0$. *We need to find* $\delta > 0$ *such that if* $|x - x_0| < \delta$ *then* $|F(x) - F(x_0)| < \varepsilon$. *We have*

$$|F(x) - F(x_0)| = |\int_0^1 (f(x+t) - f(x_0+t))dt| \leq \int_0^1 |f(x+t) - f(x_0+t)|dt.$$

Since f is continuous on $\mathbb{R}$, *it is uniformly continuous on the compact set* $[x_0 - 2, x_0 + 2]$. *Therefore, we may choose a* $\delta < 1$ *such that if* $a, b \in [x_0 - 2, x_0 + 2]$

with $|a-b|<\delta$ then $|f(a)-f(b)|<\varepsilon$. Now if $|x-x_0|<\delta$ then for all $t\in[0,1]$, we have $x+t, x_0+t\in[x_0-2, x_0+2]$ and so if $|x-x_0|<\delta$ then $|f(x+t)-f(x_0+t)|<\varepsilon$. So $|F(x)-F(x_0)|<\int_0^1 \varepsilon dt=\varepsilon$ as required.

(5) **(20 marks)** Consider the power series

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{(4k+1)!}x^{4k+1}.$$

(a) Prove that the series converges absolutely and uniformly on $[-a,a]$ for all $a>0$. Deduce that this power series defines a C^∞ function $f:\mathbb{R}\to\mathbb{R}$.

The easiest way to prove this is to observe that for all $x\in[-a,a]$ and all $r<s$, we have $\sum_{k=r}^{s}|\frac{x^{4k+1}}{(4k+1)!}|=\sum_{k=r}^{s}\frac{|x|^{4k+1}}{(4k+1)!}\le\sum_{k=r}^{s}\frac{a^{4k+1}}{(4k+1)!}\le\sum_{b=4r+1}^{4s+1}\frac{a^b}{b!}$. The expression $\sum_{b=4r+1}^{4s+1}\frac{a^b}{b!}$ is the tail of a power series defining the number e^a. So given $\varepsilon>0$, there exists $M\in\mathbb{N}$ such that if $u,v>M$ then $\sum_{b=u}^{v}\frac{a^b}{b!}<\varepsilon$. Thus, $\sum_{k=r}^{s}|\frac{x^{4k+1}}{(4k+1)!}|<\varepsilon$ if $r,s>M$. So the given series converges absolutely and uniformly on $[-a,a]$ as claimed. The power series defines a C^∞ function by Theorem 7.3.4 of the textbook.

(b) Prove that

$$f(x)+f'(x)+f''(x)+f'''(x)=e^x$$

for all $x\in\mathbb{R}$.

We are allowed to differentiate the power series term-by-term at any x for which it converges. We have

$$\begin{aligned}f(x)&=x+\frac{1}{5!}x^5+\frac{1}{9!}x^9+\cdots\\ f'(x)&=1+\frac{1}{4!}x^4+\frac{1}{8!}x^8+\cdots\\ f''(x)&=0+\frac{1}{3!}x^3+\frac{1}{7!}x^7+\cdots\\ f'''(x)&=0+\frac{1}{2!}x^2+\frac{1}{6!}x^6+\cdots\end{aligned}$$

Absolute convergence guarantees that we may rearrange these series as we wish, and the sum is the series for e^x, as required.

(c) Show that $\lim_{x\to\infty}f(x)=\infty$ and $\lim_{x\to-\infty}f(x)=-\infty$.

If $x > 0$ then $f(x) \geq x$ and if $x < 0$ then $f(x) \leq -x$ since the series for f contains only odd powers of x. This is enough to show the required properties.

(d) Show that $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

To show that f is onto, if $c \in \mathbb{R}$ then there exists $a < 0$ with $f(a) < c$ and there exists $b > 0$ with $f(b) > c$, because we showed in the previous part that f in unbounded from above and from below. Since f is continuous, the intermediate value theorem now implies that there exists z with $a < z < b$ and $f(z) = c$. To show that f is one-to-one, observe from above that $f'(x) \geq 1 > 0$ for all x, since the expression for $f'(x)$ contains only even powers of x. So f is strictly increasing. Therefore, if $a < b$ then $f(a) < f(b)$, which implies that f is one-to-one. Thus, f is a bijection.

[END.]