

413: PROBLEMS ON SETS AND PROOFS (REVISION)

Here are some practice problems on the topics discussed in Lecture 1. You should already be familiar from previous courses with the techniques used in these problems. Try to write your proofs as clearly and rigorously as possible. If you have difficulty with these problems, please tell the lecturer, since there may be gaps in your knowledge which can be filled in as we go along.

- (1) Let A and B be sets and B_1, B_2 be subsets of B . Let $f : A \rightarrow B$ be a function. Show that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
- (2) Let A and B be sets and $f : A \rightarrow B$ be a function. Show that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subset A$ if and only if f is one-to-one.
- (3) (**Proof by induction**) Show that $1 + 2 + \dots + n = n(n + 1)/2$ for all $n \in \mathbb{N}$.
- (4) (**Dividing things into cases**) If a and b are sets, define

$$(a, b) := \{\{a\}, \{a, b\}\}.$$

(the symbol $:=$ is sometimes used in mathematics to mean "is defined to be equal to".) Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

- (5) (**Proof by contradiction**) Suppose we want to place $n+1$ pigeons into n pigeonholes. Show that one pigeonhole must contain at least 2 pigeons.
- (6) (**Combination of the above**) Define $S_0 = \emptyset$, the empty set, and for $n \in \mathbb{N}$ let $S_n = \{S_{n-1}\}$, the set whose unique element is S_{n-1} . Prove $\forall m \in \mathbb{N} \cup \{0\} \forall n \in \mathbb{N} \cup \{0\}$, if $m \neq n$ then $S_m \neq S_n$.
- (7) (**Hempel's paradox**) An ornithologist is studying the hypothesis: "all crows are black". Every day she goes out with a notebook and pencil and every time she sees a black crow, she makes a note of it, since it is clearly evidence in support of the hypothesis. One day she sees a purple llama. What should she do?