## 413: PROBLEM SET 4. DUE THURSDAY 28 FEBRUARY

(1) Let $\{x_n\}$ be a Cauchy sequence of rational numbers. Since $\mathbb{Q} \subset \mathbb{R}$, we may regard $\{x_n\}$ as a Cauchy sequence of real numbers. Since every Cauchy sequence of real numbers converges, this sequence has a limit. Prove that this limit is the real number defined as the equivalence class of the original sequence $\{x_n\}$.

(2) Since $\mathbb{Q}$ is a countable set, we may write down its elements as a sequence $q_1, q_2, q_3, \ldots$ (this is called an *enumeration* of $\mathbb{Q}$). What are the limit points of such a sequence? Does the set of limit points depend on the choice of enumeration?

(3) Let $\{x_n\}$ be any sequence of real numbers and let $\{a_n\}$ be a sequence which converges to 0. Prove that the set of limit points of the sequence $\{x_n + a_n\}$ is the same as the set of limit points of $\{x_n\}$.

(4) Find all the limit points in $\mathbb{R} \cup \{\pm\infty\}$ of the following sequences. Identify the limsup and liminf.
   (a) $\{(-1)^n + \frac{1}{n} + 2^{-n}\}$.
   (b) $\{(-2)^n\}$.

(5) Section 3.1.3, #5.

(6) Section 3.1.3, #9.

(7) A subset $S \subset \mathbb{R}$ is an *interval* if for all $x, y \in S$ and for all $z \in \mathbb{R}$, if $x < z < y$ then $z \in S$. Prove that the intersection of an arbitrary collection of intervals is always an interval.

(8) Section 3.2.3 #1.

(9) Section 3.2.3 #7. (Note: limit-point is here a synonym for cluster point; see p. 92 of the textbook.)