413: PROBLEM SET 5. DUE THURSDAY 6 MARCH

(1) Let $X \subset \mathbb{R}$. A point $x \in X$ is called an *isolated point* of X if there exists an open set U such that $U \cap X = \{x\}$. Can an open set have an isolated point? Can a closed set have one?

(2) Let $X \subset \mathbb{R}$ and $x \in X$ be an isolated point. Let $f : X \to \mathbb{R}$ be any function. Show that f is continuous at x.

(3) Consider the following collection of open intervals in $\mathbb{R}$:

$$\mathcal{U} = \{(q - \frac{1}{n}, q + \frac{1}{n}) : q \in \mathbb{Q}, n \in \mathbb{N}\}.$$

(a) Prove that $\mathcal{U}$ is countable.

(b) Prove that every open subset of $\mathbb{R}$ may be expressed as a union of intervals from $\mathcal{U}$.

(c) Let $\mathcal{V}$ be the collection of all open subsets of $\mathbb{R}$. Prove that $|\mathcal{V}| = |\mathbb{R}|$. (Hint: you may use the fact, proved in lectures, that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.)

(4) If $X \subset \mathbb{R}$, prove that the closure of X is equal to the intersection of all the closed sets containing X.

(5) Recall that an *interval* in $\mathbb{R}$ is a subset I of $\mathbb{R}$ such that for all $x, y \in I$ and all $z \in \mathbb{R}$ with $x < z < y$, we have $z \in I$. Prove that if I is a bounded interval then $(x, y) \subset I$ where $x = \inf I$ and $y = \sup I$. Deduce that I must be one of the following four intervals: $(x, y), (x, y], [x, y)$ or $[x, y]$.

(6) Section 3.2.3 #6.

(7) Section 3.3.1 #6.

(8) Section 3.3.1 #8.

(9) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Give a precise definition of the statement

$$\lim_{x\to\infty} f(x) = \infty.$$

Show using your definition that $\lim_{x\to\infty} f(x) = \infty$ where $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^2$.