

413: PROBLEM SET 5. DUE THURSDAY 27 MARCH

- (1) Section 4.1.5 #8.
- (2) Section 4.2.4 #11.
- (3) In this problem, we will prove that if $x > 0$ and $n \in \mathbb{N}$, then x has a n^{th} root in \mathbb{R} .
 - (a) Show that if $x > 0$ then there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha^n < x < \beta^n$.
 - (b) Let $n \in \mathbb{N}$. Show that the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = x^n$ is continuous.
 - (c) Show that if $x > 0$ and $n \in \mathbb{N}$ then there exists $y > 0$ with $y^n = x$.
- (4) Show that if $x > 0$ then there is a *unique* $y > 0$ such that $y^2 = x$.
- (5) Given $x > 0$, we denote by \sqrt{x} the unique real number $y > 0$ such that $y^2 = x$. Consider the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Show that f is continuous. Show that f is differentiable. Is f uniformly continuous?
- (6) Suppose two monasteries, monastery A and monastery B, are joined by exactly one path, denoted AB, which is 15 km long.
 - (a) One morning Brother Albert (a monk) sets out from monastery A to monastery B. He leaves monastery A at 8am and arrives at monastery B at 8pm. The next day, he leaves monastery B at 8am and arrives back at monastery A at 8pm. On both journeys, he may have stopped occasionally to rest, or even walked in the wrong direction for some of the time. Prove, using the intermediate value theorem, that there is some point x on the path AB such that Brother Albert was at x at exactly the same time on both days.
 - (b) Suppose Brother Albert had lingered an extra hour in monastery B and left at 9am instead, but still arrived back at A at 8pm. Would this change your answer to Part (a)?