413: PROBLEM SET 2. DUE TUESDAY 12 FEBRUARY

Exercises on ordered fields:

Let F be an ordered field with the set of positive elements $F^+ \subset F$. Recall that for $x \in F$, we define the *absolute value* $|x|$ (sometimes pronounced as "mod x") as follows:

$$|x| = \begin{cases} x & x \in F^+ \\ -x & x \notin F^+ \end{cases}$$

Recall also that we define $a \leq b$ if $b + (-a) \in F^+ \cup \{0\}$. In the following exercises, you may use all theorems about ordered fields which were proved in lectures, but state clearly any theorems which you use.

(1) Show carefully that for all $a, b, c \in F$, if $a \leq b$ then $a + c \leq b + c$.

(2) Show carefully that for all $a \in F$, $a \leq |a|$.

(3) Have a look in some books and find two examples of an ordered field other then $\mathbb{Q}$ or $\mathbb{R}$, including one which does not satisfy the archimedean property.

Exercises on Chapter 2 material:

(4) Let $\{x_n\}$ be a sequence of rational numbers which converges to $L \in \mathbb{Q}$ and to $M \in \mathbb{Q}$. Show that $L = M$.

(5) Let $\{x_n\}$ and $\{y_n\}$ be sequences of rational numbers and suppose $\{x_n\}$ and $\{y_n\}$ both converge to the limit $L \in \mathbb{Q}$. Show that $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy sequences.

(6) Section 2.1.3, #1.

(7) Section 2.1.3, #8.

(8) Section 2.2.4 #4.

(9) Section 2.2.4 #5.