MATH 413 HONORS INTRODUCTION TO ANALYSIS I
PRELIM 1.
TUESDAY 11 MARCH 2008

(Note: attempt all questions. You have 70 minutes. Good luck!)

(1) **(9 marks)** Let $X = [0,1] \cup \{3\} \subset \mathbb{R}$. State whether the following statements about X are true or false and give a brief reason in each case.

(a) X is bounded.

(b) X can be written as an intersection of countably many open sets.

(c) There is a point $x_0 \in X$ at which the function $f(x) = x^4 - 3x^2 + 4$ achieves its infimum on X (that is, $f(x_0) = \inf\{f(x) : x \in X\}$).

(2) **(25 marks)** Let $\{x_n\}$ be a sequence of real numbers.

(a) **(3 marks)** Define what it means for $\{x_n\}$ to converge to a limit $L \in \mathbb{R}$.

(b) **(10 marks)** Show that if $\{x_n\}$ converges, then $\{x_n\}$ is bounded.

(c) **(6 marks)** Prove the following theorem using any method you wish:

Theorem: If $\{x_n\}$ converges to L then $\{x_n^4 - 3x_n^2 + 4\}$ converges to $L^4 - 3L^2 + 4$.

(d) **(6 marks)** Show that the converse of the theorem in part (c) is false.

(3) **(16 marks)** Sally took an analysis exam and in the final question was asked to prove the following theorem:

Theorem. *If* $s = \sup\{x \in \mathbb{Q} : x^2 < 2\}$ *then* $s^2 \geq 2$.

Her proof began as follows:

Proof: Suppose for a contradiction that $s^2 < 2$. Let $\varepsilon = 2 - s^2 > 0$. By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{2s}{n} < \varepsilon/2$. Choose such an n which is large enough so that $\frac{1}{n^2} < \varepsilon/2$. Then $\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \varepsilon \ldots$

(a) **(9 marks)** Unfortunately, Sally ran out of time here. Finish her proof of the theorem.

(b) **(7 marks)** Prove that $s^2 = 2$. [**END OF PAPER**]