MATH 2220 FINAL EXAM

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) Let D be the region in $\mathbb{R}^2$ defined by $1 \leq x^2 + y^2 \leq 4$ and $y \geq 0$.

(a) Calculate

$$\iint_D x^2 dA$$

by using polar coordinates. (Recall that $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.)

(b) Calculate the line integral

$$\int_{\partial D} (e^{x^2} + x^2 y)dx + x^3 dy$$

where ∂D has the anticlockwise orientation.

(2) (a) Find and classify the critical points of the function

$$f(x, y) = 2y^2 + 3xy - 2x^2.$$

(b) Find the absolute maximum and minimum of the function $g(x, y) = 6 + 4x + 4y$ on the disc $x^2 + y^2 \leq 1$.

(3) The surface of a mountain is given by

$$z = 9 - x^2 - y^2$$

and $z \geq 0$. A mountaineer, Klaus, is standing on the mountain at the point $(1, 1, 7)$.

(a) Find a normal vector to the mountain at the point $(1, 1, 7)$.

(b) Find the tangent plane P to the mountain at the point $(1, 1, 7)$.

(c) Find the distance from the plane P to the point $(0, 0, 9)$.

(d) Klaus wants to toboggan down the mountain in the steepest direction possible. In which direction should he proceed? [TURN OVER.]

(4) (a) The mountain in the previous question is to be bulldozed to make way for a plain. The contractor wants to know how much earth will have to be moved. Use cylindrical coordinates to find the total volume of the mountain.

(b) A tombstone T in the shape of the cuboid $[0,1] \times [0,\frac{1}{2}] \times [0,2]$ is made of marble whose density at the point (x,y,z) is $f(x,y,z) = 3 - z$. Calculate the mass

$$\iiint_T f(x,y,z)dV$$

of the tombstone.

(5) (a) Using Green's Theorem, show that the area enclosed by a simple closed curve C in $\mathbb{R}^2$ is

$$-\frac{1}{2}\oint_C (ydx - xdy)$$

where C is oriented anticlockwise.

(b) The *cardioid* is a simple closed curve with the parametrization

$$x = \cos(t) + \cos^2(t)$$

$$y = \sin(t) + \sin(t)\cos(t)$$

with $0 \leq t \leq 2\pi$. Find the area enclosed by the cardioid.

(6) Define a vector field $\mathbf{F}$ on $\mathbb{R}^3$ by

$$\mathbf{F}(x,y,z) = (y^2e^z, 2xye^z, xy^2e^z).$$

(a) Show that $\mathbf{curl}(\mathbf{F}) = \mathbf{0}$.

(b) Use a systematic method to find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

(c) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the path given by the parametrization

$$c(t) = ((t-1)e^{t^2}, \sin(\pi t/2), t)$$

with $1 \leq t \leq 2$.

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(7) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = e^x \cos(y).$$

(a) Let S be the surface in $\mathbb{R}^3$ given by the equation $z = x^2$ with $0 \leq x \leq 1$, $0 \leq y \leq \pi/4$. Set up an integral for the flux

$$\iint_S \nabla f \cdot d\mathbf{S}.$$

where S is oriented according to the normal vector with positive z–component. Be sure to include all limits of integration, but do not evaluate the integral.

(b) If M is a solid region in $\mathbb{R}^3$ to which the Divergence Theorem applies, show that

$$\iint_{\partial M} \nabla f \cdot d\mathbf{S} = 0.$$

(8) Let $\mathbf{E}$ be the vector field

$$\mathbf{E}(x, y, z) = (3ty + 1, t^2 x + z, e^t y)$$

on $\mathbb{R}^3$. The vector $\mathbf{E}(x, y, z)$ depends on a parameter t.

(a) Find $\mathbf{curl}(\mathbf{E})$ (your answer should depend on t).

(b) Use Stokes' Theorem to calculate the line integral

$$\int_C \mathbf{E} \cdot d\mathbf{r}$$

where C is the boundary of the parallelogram with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(2, 0, 0)$ and $(3, 0, 1)$. You may choose either orientation for C.

(c) Your friend Max claims that there is another field $\mathbf{B}$, depending on t, such that

$$\mathbf{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\mathbf{curl}(\mathbf{B}) = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

for some constant $c \neq 0$. Show that this cannot be true.

[END.]