MATH 2220 FINAL (PRACTICE)

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. **The exam is printed on both sides of the paper.**

Good luck!

(1) Calculate:

(a)

$$\int_0^1 \int_0^x \sin(y) dy dx.$$

(b)

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{3y^2}{x^7+1} dx dy.$$

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x(y + 1).$$

(a) Find all critical points of $f(x, y)$ and determine their nature.

(b) Find the absolute maximum and minimum of $f(x, y)$ on the disc $x^2 + y^2 \le 3$.

(3) Let $P(x, y) = \frac{-y}{x^2+y^2}$ and $Q(x, y) = \frac{x}{x^2+y^2}$. Let $\mathbf{F}(x, y) = (P, Q)$, a vector field on $\mathbb{R}^2$ which is not defined at the point $(0, 0)$.

(a) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the circle $x^2 + y^2 = \varepsilon^2$, oriented anticlockwise.

(b) Let $\Omega \subset \mathbb{R}^2$ be a region with $(0, 0)$ not in Ω and $(0, 0)$ not on $\partial\Omega$. Calculate

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(c) Let C be any simple closed curve such that $(0, 0)$ is not on C and $(0, 0)$ lies *inside* the region enclosed by C. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(4) Let S be the surface $x^2 + y^2 + z^4 = 1$.

(a) Find the tangent plane P_1 to S at $(0, 0, 1)$.

(b) Find the tangent plane P_2 to S at $(1, 0, 0)$.

(c) Find the intersection of the planes P_1 and P_2.

(5) Let Ω be the region enclosed by the plane $2x + 2y + z = 7$ and the paraboloid $z = x^2 + y^2$.

(a) Sketch Ω.

(b) Set up a triple integral for the volume of Ω. Be sure to write all the limits of integration, but do not attempt to evaluate the integral.

(c) Calculate the volume of Ω by using cylindrical coordinates.

(6) Let S be the surface in $\mathbb{R}^3$ given by the parametrization

$$\Phi(u, v) = \left(u, \frac{\cos(v)}{u^2}, \frac{\sin(v)}{u^2}\right)$$

with $1 \leq u \leq 3$, $0 \leq v \leq 2\pi$.

(a) Find a unit normal vector to S at a point $\Phi(u, v)$.

(b) Find the tangent plane to S at the point $(2, \frac{1}{4}, 0)$.

(c) Find the surface integral

$$\iint_S f(x, y, z) dS$$

where

$$f(x, y, z) = \frac{1}{\sqrt{4 + 2x^6}}.$$

(7) Define a vector field $\mathbf{F}$ on $\mathbb{R}^3$ by

$$\mathbf{F} = (3x + 5y + 7z, z^2 e^z + x^2, z^2 - 1)$$

(a) Calculate the outward flux of $\mathbf{F}$ through the cube $[0, 1] \times [0, 1] \times [0, 1]$.

(b) Calculate the outward flux of $\mathbf{F}$ through the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1.$$

(8) A beaver has built a dam D in the shape of the rectangle $x = 1$, $-2 \leq y \leq 2$, $0 \leq z \leq 2$ in $\mathbb{R}^3$. The flow of water in $\mathbb{R}^3$ at time t is given by the vector field

$$\mathbf{F}_t = (tx + e^{-t} y, 2ty, \frac{1}{z^2 + 1}).$$

(a) Calculate the flux

$$\iint_D \mathbf{F}_t \cdot d\mathbf{S}$$

of $\mathbf{F}_t$ through the dam at time t, where D is oriented with respect to the normal vector with positive x–component.

(b) The dam will burst if the flux of $\mathbf{F}_t$ through D reaches a value of 32. At what time t will the dam burst?

(c) Now suppose that the flow of water is instead given by the field $\frac{\partial \mathbf{F}_t}{\partial t}$. Will the dam burst now (ie. will the flux of $\frac{\partial \mathbf{F}_t}{\partial t}$ ever reach a value of 32)?

[END OF PAPER.]

EXTRA QUESTIONS

Note: some of these are harder than what is likely to be on the exam. Also, see the textbook for more practice problems.

(1) Derive Green's Theorem from the Divergence Theorem.

(2) Let S be the surface given by the equation $x = z^2 + y^2z^2 + y^2$, $x^2 + y^2 \leq 1$.

(a) Set up an integral for the surface area of S. Include all limits of integration.

(b) Set up an integral for the flux of the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ through S with either orientation. Include all limits of integration.

(3) Let B be a solid region in $\mathbb{R}^3$. Show that the volume of B is

$$\frac{1}{3}\iint_{\partial B} (x, y, z) \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit outward normal to B.

(4) Let $f(x, y, z)$ be a C^2 function. Show that

$$\text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

(5) Let P be a pipe whose ends are the circles in the xz–plane with centers $(0, 0)$ and $(9, 9)$ respectively, and radii 1. Let $\mathbf{F}$ be the vector field $(yx, yx^2 + ye^z, 3y)$.

(a) Calculate $\mathbf{curl}(\mathbf{F})$.

(b) Calculate the outward flux of $\mathbf{curl}(\mathbf{F})$ through P.

(6) Let f and g be C^1 functions on $\mathbb{R}^3$. Show that

$$\mathbf{curl}(f\nabla g) = \nabla f \times \nabla g.$$

(7) Let C be a simple closed curve in $\mathbb{R}^3$ which is the boundary of a surface S. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function. Show that

$$\oint_C f\nabla f \cdot d\mathbf{r} = 0.$$

(8) Bob says that $\mathbf{curl}(\text{div}(\mathbf{F}))$ is always zero, where $\mathbf{F}$ is a vector field. Explain why this statement is very wrong.

(9) Let B be the cube $[0,2] \times [0,2] \times [0,2]$ with the cube $[0,1] \times [0,1] \times [0,1]$ cut out. Let $\mathbf{F}(x,y,z) = (x,y,z)$. Calculate the outward flux of $\mathbf{F}$ through ∂B.

(10) If $\mathbf{F}$ is a vector field, is $\mathbf{curl}(\mathbf{curl}(\mathbf{F}))$ always zero?

(11) Use triple integration to derive the formula

$$V = \pi \int_a^b (f(x))^2 dx$$

for the volume of the solid obtained by rotating the graph of $y = f(x)$ between $x = a$ and $x = b$ about the x–axis. (Hint: use cylindrical coordinates with x playing the role that z usually plays).

(12) Consider the vector fields $\mathbf{F}_1 = (y, -x, 0)$ and $\mathbf{F}_2 = (y, 0, 0)$. Show that both of them have nonzero curl. Draw a picture of each of the fields and explain why they "curl", ie. why there are vortices. Draw a picture of the field $(x^2, y, 0)$ and explain why it does not curl.

(13) Let S be the surface consisting of the portion of the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 1$ together with the disc $x^2 + y^2 \le 1$, $z = 0$ in the xy–plane.

Suppose that the region between the planes $z = 0$ and $z = 1$ is filled with a fluid whose velocity varies with time. The velocity of the fluid at the point (x, y, z) at time t, $0 \le t \le 1$, is given by the vector field $\mathbf{curl}(\mathbf{F})$ where

$$\mathbf{F} = (-ty, tx, (z-1)e^{2t\sin(xz)}).$$

(a) Use Stokes' Theorem to show that the outward flux $\phi(t)$ of $\mathbf{F}$ through S at time t is

$$\phi(t) = \iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S} = 2\pi t.$$

(b) Calculate how much fluid flows outward through the surface S between time $t = 0$ and time $t = 1$.

(c) Would the answers to parts (a) and (b) change if S was replaced by another surface S' with $\partial S' = \partial S$?