MATH 2220 FINAL EXAM: SOLUTIONS

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) Let D be the region in $\mathbb{R}^2$ defined by $1 \leq x^2 + y^2 \leq 4$ and $y \geq 0$.

(a) Calculate

$$\iint_D x^2 dA$$

by using polar coordinates. (Recall that $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.)

In polar coordinates, the given region is described by $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$. So

$$\iint_D x^2 dA = \int_1^2 \int_0^\pi r^2 \cos^2(\theta) r d\theta dr = \int_1^2 r^3 dr \cdot \int_0^\pi \frac{1}{2}(1 + \cos(2\theta)) d\theta.$$

The integral of $\cos(2\theta)$ from 0 to π is 0, so we get

$$\left. \frac{r^4}{4} \right|_1^2 \cdot \frac{\pi}{2} = \frac{15\pi}{8}.$$

(b) Calculate the line integral

$$\int_{\partial D} (e^{x^2} + x^2 y) dx + x^3 dy$$

where ∂D has the anticlockwise orientation.

Using Green's Theorem,

$$\int_{\partial D} (e^{x^2} + x^2 y) dx + x^3 dy = \iint_D \left(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(e^{x^2} + x^2 y) \right) dA = \iint_D 2x^2 dA.$$

This is twice the answer to part (a), which is $\frac{15\pi}{4}$.

(2) (a) Find and classify the critical points of the function

$$f(x, y) = 2y^2 + 3xy - 2x^2.$$

The critical points are those where $\nabla f = 0$. Now, $\nabla f = (3y - 4x, 4y + 3x)$, so (x, y) is a critical point if and only if

$$\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The only solution is $x = y = 0$, so this is the only critical point. We can determine its nature using the Hessian test. In this case, the Hessian matrix is

$$H = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

We see that $\det(H) = -25 < 0$ and thus $(0, 0)$ is a saddle.

(b) Find the absolute maximum and minimum of the function $g(x, y) = 6 + 4x + 4y$ on the disc $x^2 + y^2 \leq 1$.

We look at the interior of the disc first. For (x, y) to be a critical point of f, we must have $\nabla f(x, y) = \mathbf{0}$. But $\nabla f = (4, 4)$, which is never the zero vector. So there are no critical points, and therefore the maximum and minimum must occur on the boundary. This is the circle $x^2 + y^2 = 1$. So we need to find the maximum and minimum values of $6 + 4x + 4y$ subject to $h(x, y) = x^2 + y^2 = 1$. We can do this using Lagrange multipliers. The equation $\nabla g = \lambda \nabla h$ yields

$$(4, 4) = \lambda(2x, 2y).$$

Now, λ cannot be 0, so we have $x = y = \frac{4}{2\lambda}$. Plugging into $x^2 + y^2 = 1$ gives $x = y = \pm\frac{1}{\sqrt{2}}$. Comparing the values at these two points, we see that the maximum of g is attained at $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and is $6 + \frac{8}{\sqrt{2}}$ and the minimum is attained at $(x, y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and is $6 - \frac{8}{\sqrt{2}}$.

(3) The surface of a mountain is given by

$$z = 9 - x^2 - y^2$$

and $z \geq 0$. A mountaineer, Klaus, is standing on the mountain at the point $(1, 1, 7)$.

(a) Find a normal vector to the mountain at the point $(1, 1, 7)$.

The equation of the mountain is $9 - x^2 - y^2 - z = 0$, so a normal vector is $\nabla(9 - x^2 - y^2 - z) = (-2x, -2y, -1)$. At $(1, 1, 7)$, this is $(-2, -2, -1)$.

(b) Find the tangent plane P to the mountain at the point $(1, 1, 7)$.

This is the plane passing through $(1, 1, 7)$ and normal to $(-2, -2, -1)$. The equation of this plane is

$$-2(x - 1) - 2(y - 1) - (z - 7) = 0$$

which simplifies to

$$2x + 2y + z - 11 = 0.$$

(c) Find the distance from the plane P to the point $(0, 0, 9)$.

The formula for the distance from the point (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Here, this is given by

$$\frac{|0 + 0 + 9 \cdot 1 + (-11)|}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}.$$

(d) Klaus wants to toboggan down the mountain in the steepest direction possible. In which direction should he proceed?

He should go in the direction of $-\nabla f$ where the equation of the mountain is $z = f(x, y)$. Therefore, he should go in the direction $-\nabla(9-x^2-y^2)|_{(1,1)} = (2, 2)$, or northeast.

(4) (a) The mountain in the previous question is to be bulldozed to make way for a plain. The contractor wants to know how much earth will have to be moved. Use cylindrical coordinates to find the total volume of the mountain.

The base of the mountain is the circle $x^2 + y^2 = 9$ in the xy–plane, so in cylindrical coordinates the mountain is described by $0 \le r \le 3$, $0 \le \theta \le 2\pi$ and $0 \le z \le 9 - r^2$. Therefore, the volume is

$$\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r dz dr d\theta.$$

This equals $2\pi \int_0^3 (9r - r^3)dr$, which gives

$$2\pi \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^3 = 2\pi \left[\frac{81}{2} - \frac{81}{4}\right] = \frac{81\pi}{2}.$$

(b) A tombstone T in the shape of the cuboid $[0,1] \times [0, \frac{1}{2}] \times [0,2]$ is made of marble whose density at the point (x,y,z) is $f(x,y,z) = 3 - z$. Calculate the mass

$$\iiint_T f(x,y,z)dV$$

of the tombstone.

$$\iiint_T f(x,y,z)dV = \int_0^1 \int_0^{1/2} \int_0^2 (3-z)dzdydx = \frac{1}{2}[3z - \frac{z^2}{2}]_0^2 = 2.$$

(5) (a) Using Green's Theorem, show that the area enclosed by a simple closed curve C in $\mathbb{R}^2$ is

$$-\frac{1}{2}\oint_C (ydx - xdy)$$

where C is oriented anticlockwise.

According to Green's Theorem,

$$-\frac{1}{2}\oint_C (ydx - xdy) = -\frac{1}{2}\iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right) dA = \iint_D dA$$

where D is the region enclosed by C. This is precisely the area of D, as required.

(b) The *cardioid* is a simple closed curve with the parametrization

$$x = \cos(t) + \cos^2(t)$$

$$y = \sin(t) + \sin(t)\cos(t)$$

with $0 \le t \le 2\pi$. Find the area enclosed by the cardioid.

The integral $\oint_C ydx - xdy$ equals

$$\int_0^{2\pi} (\sin(t) + \sin(t)\cos(t))(-\sin(t) - 2\cos(t)\sin(t))dt$$
$$- (\cos(t) + \cos^2(t))(\cos(t) + \cos^2(t) - \sin^2(t))dt.$$

Multiplying everything out, we get

$$\int_0^{2\pi} -\sin^2(t) - 2\cos(t)\sin^2(t) - \sin^2(t)\cos(t) - 2\sin(t)\cos^2(t)$$
$$- \cos^2(t) - \cos^3(t) + \sin^2(t)\cos(t) - \cos^3(t) - \cos^4(t) + \sin^2(t)\cos^2(t)dt$$

Gathering terms, we get

$$\int_0^{2\pi} -1 - 2\sin^2(t)\cos(t) - 2\cos^3(t) - \sin^2(t)\cos^2(t) - \cos^4(t)dt.$$

This equals

$$\int_0^{2\pi} -1 - 2\sin^2(t)\cos(t) - 2\cos(t)(1-\sin^2(t)) - \sin^2(t)\cos^2(t) - \cos^2(t)(1-\sin^2(t))dt.$$

Combining some terms, we get

$$\int_0^{2\pi} -1 - 4\sin^2(t)\cos(t) - 2\cos(t) - \cos^2(t)dt$$

This yields

$$-2\pi - 4\left[\frac{1}{3}\sin^3(t)\right]_0^{2\pi} - 0 - \int_0^{2\pi} \cos^2(t)dt = -2\pi - \pi = -3\pi.$$

By part (a), the enclosed area is then $\frac{-1}{2}(-3\pi) = \frac{3\pi}{2}$.

(6) Define a vector field $\mathbf{F}$ on $\mathbb{R}^3$ by

$$\mathbf{F}(x,y,z) = (y^2e^z, 2xye^z, xy^2e^z).$$

(a) Show that $\mathbf{curl}(\mathbf{F}) = \mathbf{0}$.

$$\mathbf{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y^2e^z & 2xye^z & xy^2e^z \end{vmatrix} = (0,0,0).$$

(b) Use a systematic method to find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

We know there is such an f because the field is conservative. To find f, first put

$$\frac{\partial f}{\partial x} = y^2e^z.$$

This implies that $f = xy^2e^z + \phi(y,z)$ where ϕ is some function of y and z.

Now, $\frac{\partial f}{\partial y} = 2xye^z + \frac{\partial \phi}{\partial y} = 2xye^z$. This implies that $\frac{\partial \phi}{\partial y} = 0$. So $\phi(y,z) = \psi(z)$.

Now, $\frac{\partial f}{\partial z} = xy^2e^z + \psi'(z) = xy^2e^z$. So $\psi'(z) = 0$ and $\psi(z)$ is just a constant. Therefore, a potential function is $f = xy^2e^z$.

(c) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the path given by the parametrization

$$c(t) = ((t-1)e^{t^2}, \sin(\pi t/2), t)$$

with $1 \le t \le 2$.

$\mathbf{F} = \nabla f$, so the value of the integral is $f(c(2)) - f(c(1)) = f(e^4, 0, 2) - f(0, 1, 1) = 0 - 0 = 0$.

(7) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = e^x \cos(y).$$

(a) Let S be the surface in $\mathbb{R}^3$ given by the equation $z = x^2$ with $0 \le x \le 1$, $0 \le y \le \pi/4$. Set up an integral for the flux

$$\iint_S \nabla f \cdot d\mathbf{S}.$$

where S is oriented according to the normal vector with positive z–component. Be sure to include all limits of integration, but do not evaluate the integral.

We need to parametrize S. A parametrization is given by $(x, y, z) = \Phi(u, v) = (u, v, u^2)$ with $0 \le u \le 1$ and $0 \le v \le \pi$. Now, $\Phi_u = (1, 0, 2u)$ and $\Phi_v = (0, 1, 0)$, so

$$\Phi_u \times \Phi_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 0 \end{vmatrix} = (-2u, 0, 1).$$

This has positive z–component.

The field $\nabla f = (e^x \cos(y), -e^x \sin(y), 0)$ and so $\nabla f(\Phi(u, v)) = (e^u \cos(v), -e^u \sin(v), 0)$. Therefore, the flux is

$$\int_0^1 \int_0^{\pi/4} -2ue^u \cos(v) dv du.$$

(b) If M is a solid region in $\mathbb{R}^3$ to which the Divergence Theorem applies, show that

$$\iint_{\partial M} \nabla f \cdot d\mathbf{S} = 0.$$

The Divergence Theorem says that

$$\iint_{\partial M} \nabla f \cdot d\mathbf{S} = \iiint_M \operatorname{div}(\nabla f) dV.$$

But $\operatorname{div}(\nabla f) = \operatorname{div}(e^x \cos(y), -e^x \sin(y), 0) = e^x \cos(y) - e^x \cos(y) = 0$.

(8) Let $\mathbf{E}$ be the vector field

$$\mathbf{E}(x, y, z) = (3ty + 1, t^2 x + z, e^t y)$$

on $\mathbb{R}^3$. The vector $\mathbf{E}(x, y, z)$ depends on a parameter t.

(a) Find $\mathbf{curl}(\mathbf{E})$ (your answer should depend on t).

$$\mathbf{curl}(\mathbf{E}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 3ty + 1 & t^2 x + z & e^t y \end{vmatrix} = (e^t - 1, 0, t^2 - 3t).$$

(b) Use Stokes' Theorem to calculate the line integral

$$\int_C \mathbf{E} \cdot d\mathbf{r}$$

where C is the boundary of the parallelogram with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(2, 0, 0)$ and $(3, 0, 1)$. You may choose either orientation for C.

Call the parallelogram P. Stokes' Theorem says that

$$\int_{\partial P} \mathbf{E} \cdot d\mathbf{r} = \iint_P \mathbf{curl}(\mathbf{E}) \cdot d\mathbf{S} = \iint_P \mathbf{curl}(\mathbf{E}) \cdot \mathbf{n} dS$$

where $\mathbf{n}$ is a unit normal to P. Since P lies in the plane $y = 0$, a unit normal is $(0, 1, 0)$. But $\mathbf{curl}(\mathbf{E}) \cdot (0, 1, 0) = 0$, so the answer is zero.

(c) Your friend Max claims that there is another field $\mathbf{B}$, depending on t, such that

$$\mathbf{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\mathbf{curl}(\mathbf{B}) = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

for some constant $c \neq 0$. Show that this cannot be true.

If this were true, then we would have

$$\mathbf{curl}(\mathbf{curl}(\mathbf{E})) = \mathbf{curl}(-\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t} \mathbf{curl}(\mathbf{B}) = \frac{-1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

However, in part (a), we computed $\mathbf{curl}(\mathbf{E}) = (e^t - 1, 0, t^2 - 3t)$. This doesn't depend on x, y, z and so $\mathbf{curl}(\mathbf{curl}(\mathbf{E})) = \mathbf{0}$. But on the other hand,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = (0, 0, e^t y)$$

which is not the zero field.

[END.]