MATH 2220 HW10.

Due Wednesday 26 November (or earlier)

(1) Let a_i, b_i, c_i be real numbers. Define a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z).$$

(a) Calculate $\mathbf{curl}(\mathbf{F})$.

Recall that $\mathbf{curl}(\mathbf{F})$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

which equals $(c_2 - b_3, a_3 - c_1, b_1 - a_2)$.

(b) Calculate $\text{div}(\mathbf{F})$.

$a_1 + b_2 + c_3$.

(c) Find a vector field $\mathbf{F}$ such that $\mathbf{curl}(\mathbf{F}) = (3, 2, 1)$ and $\text{div}(\mathbf{F}) = 7$.

In view of the previous two parts, we just have to make an appropriate choice of a_i, b_i and c_i. There are lots of choices, eg. take $c_2 = 3, b_3 = 0, a_3 = 2, c_1 = 0, b_1 = 1, a_2 = 0, a_1 = 7, b_2 = c_3 = 0$. This gives

$$\mathbf{F}(x, y, z) = (7 + 2z, x, 3y).$$

(2) Let C be the boundary of the square in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 1)$ and $(1, 1, 1)$, oriented in the direction $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 1) \to (0, 1, 1) \to (0, 0, 0)$.

Let

$$\mathbf{F}(x, y, z) = \sin(x^3)\mathbf{i} + \mathbf{j} + (e^{z^2} + 3y)\mathbf{k}.$$

(a) Calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

It would be quite hard to calculate this directly, so instead we will use Stokes' Theorem. Let S be the square whose boundary ∂S is the given curve. Stokes'

Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

Now $\mathbf{curl}(\mathbf{F})$ equals

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \sin(x^3) & 1 & e^{z^2} + 3y \end{vmatrix} = 3\mathbf{i} = (3, 0, 0).$$

To find the flux of $(3, 0, 0)$ through S, we should first parametrize S. We observe, however, that S lies in the plane $y = z$, and thus a normal vector to S is $(0, -1, 1)$. Thus, if $\mathbf{n}$ is a unit normal to S, then $\mathbf{n}$ will be parallel to $(0, -1, 1)$ and we will have $(3, 0, 0) \cdot \mathbf{n} = 0$. So

$$\iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \mathbf{curl}(\mathbf{F}) \cdot \mathbf{n} dS = 0.$$

(b) Does there exist a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$? Explain.

If $\mathbf{F} = \nabla f$ then $\mathbf{curl}(\mathbf{F}) = \mathbf{curl}(\nabla f) = \mathbf{0}$, by a theorem from the lectures. But $\mathbf{curl}(\mathbf{F}) = (3, 0, 0) \neq \mathbf{0}$. Therefore, $\mathbf{F}$ cannot be the gradient field of any function.