MATH 2220 HW11.

Due Friday 5 December

(1) Let S be the surface obtained by rotating the graph of the function

$$y = |x| + 1$$

about the x–axis between $x = -2$ and $x = 2$.

(a) Let $\mathbf{F}(x, y, z) = (0, (x^2 - 2)z, y)$. Calculate

$$\iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

with respect to either orientation.

To solve this, we should apply Stokes' Theorem, which says that

$$\iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

We are free to choose an orientation of the surface S, so let us choose the orientation so that the normal vector at a point with $z = 0$ and $y > 0$ has a positive y component. This is the "outward" orientation (you should draw a picture to visualize it). Notice that there isn't a well-defined normal vector at every point of S because S is not smooth. This is not a problem, however.

In this example ∂S consists of two circles of radius 3, one in the $x = 2$ plane and one in the $x = -2$ plane. We can parametrize the circle in the $x = 2$ plane as $(2, \sqrt{3}\cos(t), \sqrt{3}\sin(t))$, $0 \leq t \leq 2\pi$. We need to check that this circle has the correct orientation induced from the orientation we chose on S. At $t = 0$ we are at the point $(2, 1, 0)$ on the circle and moving "up", ie in the direction of increasing z. This is indeed the orientation induced by our choice of orientation of S. Again, it is a lot easier to see this from a picture!

Let us call the circle in the $x = 2$ plane C_1. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (0, 0, \sqrt{3}\cos(t)) \cdot (0, -\sqrt{3}\sin(t), \sqrt{3}\cos(t)) dt.$$

This equals $3\int_0^{2\pi} \cos^2(t) dt = 3\pi$.

Call the circle in the $x = -2$ plane C_2. This has the parametrization $(-2, \sqrt{3}\cos(t), \sqrt{3}\sin(t))$, $0 \le t \le 2\pi$, except that we have to traverse it in the opposite direction to C_1. So we get

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -3\pi.$$

Therefore,

$$\iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} = 3\pi - 3\pi = 0.$$

(b) Let $\mathbf{F}(x, y, z) = (2x, y + z^3 e^{z^3}, x)$. Calculate the outward flux

$$\iint_{\partial B} \mathbf{F} \cdot d\mathbf{S}$$

where B is the solid region bounded by S and the planes $x = 2$ and $x = -2$.

The divergence theorem tells us that

$$\iint_{\partial B} \mathbf{F} \cdot d\mathbf{S} = \iiint_B \mathrm{div}(\mathbf{F}) dV.$$

Here $\mathrm{div}(\mathbf{F}) = 2 + 1 = 3$, so $\iiint_B \mathrm{div}(\mathbf{F}) dV = 3 \iiint_B dV$. We just need to calculate the volume of B. This is twice the volume of the solid of revolution obtained by rotating the graph of $y = x + 1$, $0 \le x \le 2$, about the x–axis. By one-variable calculus, this volume is

$$\pi \int_0^2 (x+1)^2 dx = \frac{\pi}{3}(x+1)^3|_0^2 = \frac{26\pi}{3}.$$

Thus, $\iiint_B dV = 52\pi/3$ and therefore $\iint_{\partial B} \mathbf{F} \cdot d\mathbf{S} = \iiint_B \mathrm{div}(\mathbf{F}) dV = 52\pi$.

(2) Let c be a constant and let $\mathbf{E} = \mathbf{E}_t$ and $\mathbf{B} = \mathbf{B}_t$ be time-dependent vector fields defined by

$$\mathbf{E}_t = ((t+1)x, -ty, -z)$$

$$\mathbf{B}_t = \frac{1}{c^2}(0, 0, yx).$$

(a) Show that $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$

Treating t as a parameter, $\nabla \cdot \mathbf{E} = \mathrm{div}(\mathbf{E}) = (t+1) - t - 1 = 0$ and $\nabla \cdot \mathbf{B} = 0 + 0 + \frac{\partial}{\partial z}\frac{1}{c^2} yx = 0$.

(b) Show that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$.

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (t+1)x & -ty & -z \end{vmatrix} = \mathbf{0}$$

which equals $-\frac{\partial \mathbf{B}}{\partial t}$ since $\mathbf{B}$ is independent of t.

Also,

$$\nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{1}{c^2}xy \end{vmatrix} = \frac{1}{c^2}(x, -y, 0) = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

(c) Let S be any oriented surface. Show that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = (t+1)c^2 \oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} + \oint_{\partial S} yz dx.$$

One way to do this is to write

$$\mathbf{E} = (t+1)(x, -y, 0) + (0, y, -z) = (t+1)\frac{\partial \mathbf{E}}{\partial t} + (0, y, -z).$$

By the previous problem, this equals $(t+1)c^2(\nabla \times \mathbf{B}) + (0, y, -z)$. Therefore,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = (t+1)c^2 \iint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} + \iint_S (0, y, -z) \cdot d\mathbf{S}.$$

Observe that $\mathbf{curl}(yz, 0, 0) = (0, y, -z)$. Therefore,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = (t+1)c^2 \iint_S \mathbf{curl}(\mathbf{B}) \cdot d\mathbf{S} + \iint_S \mathbf{curl}(yz, 0, 0) \cdot d\mathbf{S}.$$

By Stokes' Theorem, this equals

$$(t+1)c^2 \oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} + \oint_{\partial S} (yz, 0, 0) \cdot d\mathbf{r} = (t+1)c^2 \oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} + \oint_{\partial S} yz dx.$$

(The final step is just a change of notation.)