MATH 2220 HW1 SOLUTIONS.

Homework 1. Due Wednesday 3 September.

(1) Section 1.3, p. 61–65

(a) # 8.

The volume is given by absolute value of the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 4 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = -1.$$

So the volume is $|-1| = 1$.

(b) # 24.

The direction vector of the line is $(3, -2, 4)$, *so the desired plane is the plane through* $(2, -1, 3)$ *orthogonal to* $(3, -2, 4)$. *This plane is given by the equation*

$$3(x-2) - 2(y+1) + 4(z-3) = 0.$$

(c) # 36.

Following the hint:

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} =$$

$$(a_1 - x, a_2 - y, a_3 - z) \cdot ((b_1 - x, b_2 - y, b_3 - z) \times (c_1 - x, c_2 - y c_3 - z))$$

In the notation of the problem this is $(A-P) \cdot ((B-P) \times (C-P))$. *The problem asks us to show that* P *lies on the plane determined by* A, B *and* C *if and only if* $(A-P) \cdot ((B-P) \times (C-P)) = 0$. *(Note that this doesn't make sense if* A, B, C *are collinear, so we must assume that they are not.) One way to see the result geometrically is that the vectors* $A-P, B-P$ *and* $C-P$ *all lie in a plane if and only if the volume of the parallelepiped determined by them is zero, and this volume is precisely* $|(A-P) \cdot ((B-P) \times (C-P))|$.

Another way to do the problem is to observe that

$$D = \begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = \begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix}$$

by subtracting the first row of the determinant from the other two rows. This is the equation of a plane (to see this, expand the determinant along the top row). Since the determinant D has value zero if (x, y, z) is set to one of A, B or C, the plane determined by $D = 0$ contains A, B and C. Again, assuming that A, B and C are not collinear, it is clear geometrically that they determine a unique plane.

(2) Find a unit vector parallel to the line of intersection of the planes $x - 2y + 5z = 2$ and $3x - y + 5z = 3$.

A vector orthogonal to the first plane is $(1, -2, 5)$ and a vector orthogonal to the second plane is $(3, -1, 5)$ (read off the coefficients). A vector pointing along the line of intersection of the planes must be orthogonal to both of these, so we can find one by taking the cross product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 5 \\ 3 & -1 & 5 \end{vmatrix}$$

which equals $-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$. We want a unit vector, so we must divide by the norm, which is $\sqrt{5^2 + 10^2 + 5^2} = \sqrt{150}$. The desired vector is therefore

$$\frac{1}{\sqrt{150}}(-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}).$$

(3) Given the points $P = (1, 2, 3)$, $Q = (3, 5, 2)$ and $R = (2, 2, 3)$ find:

(a) The area of the triangle PQR.

Subtracting P from everything does not change the area of the triangle (it just moves everything by $-P$). So we want the area of the triangle with vertices 0, $Q - P$ and $R - P$. By the lectures, we know that the area of the parallelogram determined by $Q - P$ and $R - P$ is $\|(Q - P) \times (R - P)\|$, so the desired area is *half of this.*

$$(Q-P)\times(R-P)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & -1\\ 1 & 0 & 0\end{vmatrix}=-\mathbf{j}-3\mathbf{k}$$

. So $\frac{1}{2}\|(Q-P)\times(R-P)\|=\frac{1}{2}\sqrt{1^2+3^2}=\sqrt{10}/2.$

(b) The distance from R to the line through P and Q.

There are several ways of doing it; here is one. First, we must find the line through P *and* Q*. This line is given by* $P+t(Q-P)$*,* $t\in\mathbb{R}$*. Therefore, it is* $(1,2,3)+t(2,3,-1)$*,* $t\in\mathbb{R}$*. We need to find the point on this line which is closest to* R*. The distance from a general point on the line to* R *is* $\|(1+2t,2+3t,3-t)-(2,2,3)\|=\|(-1+2t,3t,-t)\|=\sqrt{(-1+2t)^2+(3t)^2+t^2}$*. This equals* $\sqrt{1-4t+14t^2}$*. This will be mimimized by the value of* t *which minimizes* $1-4t+14t^2$*, which we can find using calculus. The desired value of* t *satisfies* $-4+28t=0$*, so* $t=4/28=1/7$*. (The second derivative test guarantees that this is indeed a minimum, as it should be by geometric intuition.) The desired distance is therefore* $\sqrt{1-\frac{4}{7}+\frac{14}{49}}$*.*

An even faster method is the following. Let the desired distance be h*. Then the area of the triangle* PQR *is* $\frac{1}{2}h\cdot dist(P,Q)$*, by the formula for the area of a triangle (half times base times height). So* $h=2\cdot Area(PQR)/dist(P,Q)=\frac{\sqrt{10}}{2\|(2,3,-1)\|}=\frac{\sqrt{10}}{\sqrt{14}}$*.*