MATH 2220 HW3.

Due Wednesday 17 September

(1) Find

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

or show that it does not exist.

The limit does not exist. For example, if we approach $(0,0)$ along the path $x=0$, the limit is 0. If we approach along the path $y=\sqrt{x}$, the limit is 1. If we approach along the path $y=0$, we get a limit of ∞. If we approach along the path $y=-x$, we get a limit of $-\infty$. Considering any two of these paths is enough to show that the limit cannot exist.

(2) Section 2.3, p. 139-141

(a) # 1(d).

By the product and chain rules for differentiation of one-variable functions, we get $\frac{\partial f}{\partial x} = 2x\log(x^2+y^2)+2x$ and $\frac{\partial f}{\partial y} = 2y\log(x^2+y^2)+2y$.

(b) # 4(b).

The domain of f is $U=\{(x,y)\in\mathbb{R}^2 : x\neq 0 \text{ and } y\neq 0\}$. The partial derivatives exist and are continuous at every point of U, so f is C^1 on U and therefore differentiable on U.

(c) # 5.

$\frac{\partial z}{\partial x}|_{(3,1,10)} = 2x|_{(3,1,10)} = 6$ and $\frac{\partial z}{\partial y}|_{(3,1,10)} = 3y^2|_{(3,1,10)} = 3$. By a formula from the lectures, the tangent plane is given by

$$6(x-3)+3(y-1)-(z-10)=0.$$

(d) # 7(c).

The matrix of partial derivatives is

$$\begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

(e) # 9.

$\frac{\partial z}{\partial x}|_{(1,1,1)} = 1$ and $\frac{\partial z}{\partial y}|_{(1,1,1)} = -1$, so the tangent plane at $(1,1,1)$ is

$$(x-1)-(y-1)-(z-1)=0$$

by the formula from lectures. This may be rewritten as $x-y-z=-1$. The z–axis is the line defined by the equations $x=y=0$, so the required point has z–coordinate $0-0-z=-1 \implies z=1$. Therefore, the plane meets the z–axis at the point $(0,0,1)$.

(3) The temperature at a point $(x,y,z) \in \mathbb{R}^3$ is given by $T(x,y,z) = e^{(10-x)^2}+e^{-y^2}+e^{-z^2}$.

(a) A flamingo flies along the path

$$f(t) = (0, t, t^2+5).$$

Find the rate of change of temperature experienced by the flamingo at $t=0$, ie. find $\frac{d}{dt}|_{t=0}(T \circ f(t))$.

By the chain rule,

$$\frac{d}{dt}|_{t=0}(T \circ f(t)) = D(T \circ f)(0) = DT(f(0))Df(0).$$

We have $f(0)=(0,0,5)$ and $Df(t) = \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix}$. So $Df(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Also, $DT(x,y,z) = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}) = (-2(10-x)e^{(10-x)^2}, -2ye^{-y^2}, -2ze^{-z^2})$. Therefore, $DT(0,0,5) = (-20e^{100}, 0, -10e^{-25})$. Putting this together, we get

$$DT(f(0))Df(0) = (-20e^{100}, 0, -10e^{-25}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.$$

(b) A goose flies along the path

$$g(t) = (t^3, \sin(t), 5e^t + t^3 + t^2 - 5t).$$

Find the rate of change of temperature experienced by the goose at $t=0$.

This is rather a difficult calculation without the chain rule. As in part (a), we have

$$\frac{d}{dt}|_{t=0}(T \circ g(t)) = D(T \circ g)(0) = DT(g(0))Dg(0).$$

But $g(0) = (0, 0, 5) = f(0)$ *and* $Dg(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = Df(0)$*. So* $DT(g(0))Dg(0) =$ $DT(f(0))Df(0)$ *which we have already computed to be* 0.

(c) Suppose that the temperature (in degrees) was given by some other function $T_1(x, y, z)$ and that the flamingo experienced a rate of change of temperature of 5 degrees per second at $t = 0$. Is it possible to determine the rate of change of temperature experienced by the goose at $t = 0$? Explain.

Provided T_1 *is a differentiable function, we know that the rate of change of temperature experienced by the goose is also* 5 *degrees per second. This is because we have shown that* $f(0) = g(0)$ *and* $Df(0) = Dg(0)$*, so that*

$$\frac{d}{dt}|_{t=0}(T_1 \circ g(t)) = D(T_1 \circ g)(0) = DT_1(g(0))Dg(0)$$

which equals

$$DT_1(f(0))Df(0) = D(T_1 \circ f)(0) = \frac{d}{dt}|_{t=0}(T_1 \circ f(t)).$$

Thus the rates of change of temperature experienced by the goose and the flamingo at $T = 0$ *are always the same.*

Note that this answer makes sense, because the two birds are at the same point at $t = 0$ *and flying with the same speed and in the same direction, so it is logical that they should experience the same rate of change of temperature at that instant.*