MATH 2220 HW4.

Due Wednesday 24 September

(1) Use the derivative to estimate the value of $\cos(0.02\cos(-0.03))$.

Let $f(x,y) = \cos(x\cos(y))$. *Let* $\mathbf{x_0} = (0,0)$. *Let* $\mathbf{x} = (0.02, -0.03)$. *We estimate* $f(\mathbf{x})$ *by using the linear approximation:*

$$f(\mathbf{x_0}) + \nabla f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0})$$

We have $f_x = -\sin(x\cos(y))\cos(y)$ *and* $f_y = \sin(x\cos(y))x\sin(y)$. *Therefore* $f_x(0,0) = 0 = f_y(0,0)$ *and* $f(0,0) = 1$. *So*

$$f(\mathbf{x_0}) + \nabla f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) = 1 + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix} = 1.$$

(2) Section 2.5 p.159-163

(a) # 11.

Using the chain rule, first calculate $D(f \circ T)(1,0)$.

$$D(f \circ T)(1,0) = Df(T(1,0))DT(1,0) = Df(1, \log\sqrt{2})DT(1,0).$$

$Df(u,v) = (-\sin u \sin v, \cos u \cos v)$ *and*

$$DT(s,t) = \begin{bmatrix} -t^2\sin(t^2 s) & -2ts\sin(t^2 s) \\ \frac{2s}{\sqrt{1+s^2}} & 0 \end{bmatrix}$$

So

$$DT(1,0) = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}$$

Therefore,

$$Df(1, \log\sqrt{2})DT(1,0) = (-\sin(1)\sin(\log\sqrt{2}), \cos(1)\cos(\log\sqrt{2})) \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}$$

which equals

$$(\sqrt{2}\cos(1)\cos(\log\sqrt{2}), 0).$$

The question just asks for $\frac{\partial(f\circ T)}{\partial s}(1,0)$, *which is the first entry of* $D(T\circ s)(1,0)$, *ie.*

$$\sqrt{2}\cos(1)\cos(\log\sqrt{2}).$$

(b) # 15.

The question is asking for

$$D(f\circ \mathbf{c})(0)$$

By the chain rule, this equals

$$Df(\mathbf{c}(0))D\mathbf{c}(0) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}|_{(0,0)}\mathbf{c}'(0).$$

This in turn equals

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(c) # 24.

The problem is wrong to use $\frac{\partial w}{\partial x}$ *for two different things. The symbol* $\frac{\partial w}{\partial x}$ *on the left hand side denotes the derivative of* $f(x,y,g(x,y))$ *with respect to* x, *while on the right hand side it denotes the derivative of* $f(x,y,z)$ *with respect to* x, *treating* z *as a constant. These two things are not the same in general (try it with some nonconstant function* g *and some* f*). This is what causes the problem.*

(3) Section 2.6 p.171-173

(a) # 4(b).

The surface is given by $g(x,y,z) = 0$ *where* $g(x,y,z) = y^2 - x^2 - 3$. *Then* $\nabla g = (-2x, 2y, 0)$. *The equation of the tangent plane is therefore*

$$(-2,4,0)\cdot(x-1, y-2, z-8) = 0.$$

(b) # 14(a).

The directional derivative is $\nabla f\cdot \mathbf{v}$ *(there is no need to normalize* $\mathbf{v}$ *since it is already a unit vector).*

$$\nabla f = (y^2+z^3, 2xy+2yz^3, 3y^2z^2+3z^2x)$$

at $(4,-2,-1)$*, this has the value* $(3,-12,24)$*. The answer is therefore* $3/\sqrt{14} - 36/\sqrt{14} + 48/\sqrt{14}$.

(c) # 15.

(4) The surface of a mountain is given by the set of points (x, y, z) in $\mathbb{R}^3$ satisfying $z = 20 - (\frac{x}{10})^2 - (\frac{y}{20})^4$ and $x, y, z \geq 0$. Klaus is at the point $(10, 20, 18)$ and he wants to toboggan down the mountain in the steepest direction possible. In which direction should he go?

Let $z = f(x, y) = 20 - (\frac{x}{10})^2 - (\frac{y}{20})^4$*. Then the direction of steepest descent is* $-\nabla f(10, 20) = -(-\frac{1}{10}.2.\frac{10}{10}, -\frac{1}{20}.4.(\frac{20}{20})^3) = (\frac{1}{5}, \frac{1}{5})$*. In other words, Klaus should go due northeast.*

(5) A 2×2 matrix of real numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be identified with the point $(a, b, c, d) \in \mathbb{R}^4$. We can therefore define a map $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(A) = A^2$, the square of the matrix A. Calculate the derivative of f at an arbitrary point $B \in \mathbb{R}^4$ (also regarded as a matrix).

Write f *as*

$$f(a, b, c, d) = (a^2 + bc, ab + bd, ac + cd, cb + d^2).$$

Then

$$Df(a, b, c, d) = \begin{bmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{bmatrix}$$