MATH 2220 HW6.

Due Wednesday 15 October

(1) Section 3.5, p. 253-255.

(a) # 8.

The inverse function theorem tells us that we need to check that the matrix

$$\begin{bmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{bmatrix}$$

is invertible at $(0,0,0)$*. Plugging in these values, the matrix becomes*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

which is invertible since it has determinant 1*. So the answer is yes, the equations do define* x, y, z *as functions of* u, v, w *near* $(0,0,0)$*.*

(b) # 12.

Let $F_1 = xy^2 + xzu + yv^2 - 3$ *and let* $F_2 = u^3yz + 2xv - u^2v^2 - 2$*. The implicit function theorem tells us that we must check that the matrix*

$$\begin{bmatrix} (F_1)_u & (F_2)_v \\ (F_2)_u & (F_2)_v \end{bmatrix}$$

is invertible. This matrix is

$$\begin{bmatrix} xz & 2yv \\ 3u^yz - 2uv^2 & 2x - 2u^2v \end{bmatrix}$$

At the given point $(1,1,1,1,1)$*, this equals*

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

which is an invertible matrix. So the answer is yes.

Differentiating $F_1 = 0$ and $F_2 = 0$ with respect to y, keeping y and z constant, yields

$$2xy + xz\frac{\partial u}{\partial y} + v^2 + 2yv\frac{\partial v}{\partial y} = 0$$

$$3u^2yz\frac{\partial u}{\partial y} + u^3z + 2x\frac{\partial v}{\partial y} - 2uv^2\frac{\partial u}{\partial y} - 2vu^2\frac{\partial v}{\partial y} = 0$$

Substituting in $(1, 1, 1, 1, 1)$ yields the two equations

$$3 + \frac{\partial u}{\partial y}|_{(1,1,1)} + 2\frac{\partial v}{\partial y}|_{(1,1,1)} = 0$$

$$\frac{\partial u}{\partial y}|_{(1,1,1)} + 1 = 0$$

The second equation gives $\frac{\partial u}{\partial y}|_{(1,1,1)} = -1$. Substituting in the first equation gives $\frac{\partial v}{\partial y}|_{(1,1,1)} = -1$ as well.

(2) Review Exercises, p. 255-259.

(a) # 5.

First we need to differentiate and set $\nabla z = 0$ to find the critical points.

$$z_x = \frac{x^3 - x^2 - 2x}{1 + 4y^2}$$

and

$$z_y = \frac{1}{12}(3x^4 - 4x^3 - 12x^2 + 18)\frac{-8y}{(1 + 4y^2)^2}.$$

Therefore, $z_x = 0$ implies $x(x^2 - x - 2) = 0 = x(x - 2)(x + 1)$ so $x = 0$ or $x = 2$ or $x = -1$. None of these values of x is a root of $3x^4 - 4x^3 - 12x^2 + 18$, and therefore, if $z_y = 0$ as well, we must have $y = 0$. So we find three critical points, namely $(0, 0), (2, 0)$ and $(-1, 0)$.

We differentiate again to find the Hessian matrix:

$$z_{xx} = \frac{3x^2 - 2x - 2}{1 + 4y^2}$$

and

$$z_{yx} = (x^3 - x^2 - 2x)\frac{-8y}{(1 + 4y^2)^2}$$

(note that this is 0 at all critical points, since $y = 0$ there). Finally,

$$z_{yy} = \frac{1}{12}(3x^4 - 4x^3 - 12x^2 + 18). - 8.(1 + 4y^2)^{-2}((1 + 4y^2) + 8y(\cdots)).$$

We obtain the Hessian matrices:

at $(0,0)$,

$$\begin{bmatrix} -2 & 0 \\ 0 & -8.18/12 \end{bmatrix}$$

so $(0,0)$ is a local maximum.

At $(-1,0)$,

$$\begin{bmatrix} 3 & 0 \\ 0 & -8.13/12 \end{bmatrix}$$

so $(0,0)$ is a saddle.

At $(2,0)$,

$$\begin{bmatrix} 6 & 0 \\ 0 & 8.14/12 \end{bmatrix}$$

so $(0,0)$ is a local minimum.

(b) # 15.

We minimize the squared distance $x^2+y^2+z^2$ from the point (x,y,z) to the origin, subject to the constraint that (x,y,z) lies on the surface $z^2-xy=1$. Lagrange multipliers tell us to look for solutions of the equation

$$(2x,2y,2z)=\lambda(-y,-x,2z)$$

Thus we get the four equations

$$2x=-\lambda y$$

$$2y=-\lambda x$$

$$2z=2\lambda z$$

$$z^2-xy=1$$

From the first two equations, we get $4x=\lambda^2 x$, whence $x=0$ or $\lambda=\pm 2$. If $x=0$ then $2y=-\lambda x=0$ so $y=0$ and $z^2-0=1$ so $z=\pm 1$. Thus, the minimum may occur at the points $(0,0,\pm 1)$. The other possibility is $\lambda=\pm 2$. Then $2z=2\lambda z$ implies $z=0$ and so $xy=-1$. So $y=-1/x$ and $2x=-2\lambda y=\lambda/x$ which implies $2x^2=\lambda\geq 0$. So $\lambda=2$ and $x=\pm 1$. Then $2y=-\lambda x=-2x$ implies $y=-x$. So we get two more candidate points, namely $(-1,1,0)$ and $(1,-1,0)$.

Evaluating the distance from the four points we have found to the origin, we find that the minimum distance occurs at the points $(0, 0, \pm 1)$*. Thus, these two points are the closest to the origin on the given surface.*

(c) # 22.

We wish to optimize the function $f(x, y) = xy - y + x - 1 = (x-1)(y+1)$ *subject to* $x^2 + y^2 \leq 2$*. First, we look for critical points in the open disc* $x^2 + y^2 < 2$*. We have* $\nabla f = (y + 1, x - 1)$ *so the only critical point is* $(1, -1)$*. This does not lie in the open disc, and therefore the maximum and minimum must occur on the boundary. Now we look at the boundary curve* $x^2 + y^2 = 2$*. We use Lagrange multipliers with the constraint* $g(x, y) = x^2 + y^2 = 2$*. The equation* $\nabla f = \lambda \nabla g$ *gives* $(y+1, x-1) = \lambda(2x, 2y)$*. Now, neither* x *nor* y *can be zero, or else* $x^2 + y^2$ *cannot equal* 2*. Therefore,* $\frac{y+1}{2x} = \frac{x-1}{2y}$*. This gives* $2y^2 + 2y = 2x^2 - 2x$*, whence* $(y - x)(y + x) = y^2 - x^2 = -x - y = -(y + x)$*. So either* $y + x = 0$*, or we can divide by* $y + x$ *to get* $y - x = -1$*. From these two possibilities, we get the points* $(1, -1), (-1, 1), (1 + \sqrt{2}, \sqrt{2})$ *and* $(1 - \sqrt{2}, -\sqrt{2})$*. Comparing the values of the function at these points gives an absolute minimum of* -4 *and an absolute maximum of* $\sqrt{2}(\sqrt{2} + 1)$*.*

(3) Find the maximum and minimum values of the function $f(x, y) = e^x + e^{-y}$ on the line segment in $\mathbb{R}^2$ joining $(-1, -1)$ to $(1, 1)$.

The line segment is a closed and bounded set, but it has no interior since it is equal to its own boundary. Usually to find the maximum and minimum we would look at the interior first, but here we can skip that step and go straight to the boundary. We can write the given line segment as $(-1 + t, -1 + t)$ *with* $0 \leq t \leq 2$*. So the problem becomes to find the maximum and minimum values of the function* $e^{t-1} + e^{1-t}$ *on the closed interval* $[0, 2]$*. We first set the derivative equal to* 0 *and get* $e^{t-1} - e^{1-t} = 0$*, whence* $e^{2(t-1)} = 1$*. This has the solution* $t = 1$*. We also need to check the values of the function at the endpoints. When* $t = 0$*, we get* $e^{-1} + e$*, when* $t = 2$ *we get* $e + e^{-1}$*, and at the point* $t = 1$*, we get* 2*. Thus the maximum is* $e + e^{-1}$ *and the minimum is* 2*.*

(4) Use Lagrange multipliers to show that the distance from the point (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Let (x, y, z) be an arbitrary point on the plane. Our aim is to minimize the square of the distance from (x, y, z) to (x_0, y_0, z_0) subject to the constraint that (x, y, z) lies in the plane. That is, we must minimize $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ subject to $Ax + By + Cz + D = 0$. We introduce a Lagrange multiplier λ and obtain the equation

$$2(x - x_0, y - y_0, z - z_0) = \lambda(A, B, C)$$

Solving this gives

$$x = x_0 + \lambda\frac{A}{2}$$
$$y = y_0 + \lambda\frac{B}{2}$$
$$z = z_0 + \lambda\frac{C}{2}$$

Substituting into the constraint $Ax + By + Cz + D = 0$ gives

$$Ax_0 + By_0 + Cz_0 + D + \frac{\lambda}{2}(A^2 + B^2 + C^2) = 0$$

and thus

$$\lambda = \frac{-2(Ax_0 + By_0 + Cz_0 + D)}{A^2 + B^2 + C^2}$$

The square of the distance from (x, y, z) to $(0, 0, 0)$ is now given by

$$(\lambda A/2)^2 + (\lambda B/2)^2 + (\lambda C/2)^2 = \frac{\lambda^2}{4}(A^2 + B^2 + C^2).$$

Substituting in the value we have found for λ gives

$$\frac{(Ax_0 + By_0 + Cz_0 + D)^2}{A^2 + B^2 + C^2}.$$

The distance itself is the square root of this, which is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

as required.