MATH 2220 HW8.

Due Wednesday 12 November

(1) Section 6.2 p. 392.

(a) # 29.

For part (a), one way to do this is to change coordinates. The easiest choice is to take spherical coordinates but modify them slightly. Thus, we take

$$x = a\rho\cos(\theta)\sin(\phi)$$
$$y = b\rho\sin(\theta)\sin(\phi)$$
$$z = c\rho\cos(\phi)$$

with $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ and $0 \le \rho \le 1$. The Jacobian of our transformation $T(\rho,\theta,\phi) = (x,y,z)$ will be the Jacobian of the usual spherical coordinate transformation but with an extra factor of abc (check). So the volume of E may be expressed as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\rho^2\sin(\phi)d\rho d\phi d\theta$$

which equals

$$abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin(\phi)d\rho d\phi d\theta = abc\text{volume(sphere)} = 4\pi abc/3.$$

For part (b), we can use the same change of coordinates again. The Jacobian is still $abc\rho^2\sin(\phi)$ but this time we want to integrate

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = \rho^2.$$

So after the change of variables, our integral becomes

$$abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\rho^2\sin(\phi)d\rho d\phi d\theta = 2\pi\int_0^1 \rho^4 d\rho\int_0^{\pi}\sin(\phi)d\phi = 4\pi abc/5.$$

(2) Section 7.2 p. 447-451.

I had meant to set 3(d). and 6, so this problem was a lot harder than intended.

(a) # 12.

The question asks us to integrate a vector field $\mathbf{F}$ on $\mathbb{R}^3$ over a square in $\mathbb{R}^2$. Presumably, it means the square in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, 0)$. However, if you took some other square, the answer would still be zero.

We can split our square into four pieces and parametrize each one separately.

$c_1(t) = (1, t, 0), -1 \leq t \leq 1.$

$c_2(t) = (t, 1, 0), -1 \leq t \leq 1.$

$c_3(t) = (-1, t, 0), -1 \leq t \leq 1.$

$c_4(t) = (t, -1, 0), -1 \leq t \leq 1.$

Let's choose to go around the square anticlockwise starting at $(-1, -1)$. The desired integral is given by $\int_{c_4} \mathbf{F} + \int_{c_1} \mathbf{F} - \int_{c_2} \mathbf{F} - \int_{c_3} \mathbf{F}$ (cf. a similar example from the lectures.) Now we have to calculate each of the four pieces separately. $\int_{c_4} \mathbf{F} = \int_{-1}^{1} (-2t, t^2, 0) \cdot (1, 0, 0) dt = \int_{-1}^{1} (-2t) dt = 0$. Next, $\int_{c_1} \mathbf{F} = \int_{-1}^{1} (2t, 1, 0) \cdot (0, 1, 0) dt = \int_{-1}^{1} 1 dt = 2$. Next, $\int_{c_2} \mathbf{F} = \int_{-1}^{1} (2t, t^2, 0) \cdot (1, 0, 0) dt = 0$. Finally, $\int_{c_3} \mathbf{F} = \int_{-1}^{1} (-2t, 1, 0) \cdot (0, 1, 0) dt = \int_{-1}^{1} dt = 2$. Adding them up yields that the integral of $\mathbf{F}$ over the boundary of the square is zero.

(b) # 18.

This one is very hard for us at the moment. The secret is that the given vector field $y\mathbf{i} + x\mathbf{j} + \mathbf{k} = (y, x, 1)$ is the gradient ∇f of the function $f(x, y, z) = xy + z$. This has to be integrated along some complicated path which starts at $(\sqrt{2\pi}, 0, 0)$ and ends at $(0, 0, 2\pi)$. The integral of ∇f over any path starting at $\mathbf{a}$ and ending at $\mathbf{b}$ is just $f(\mathbf{b}) - f(\mathbf{a})$. The answer is therefore

$$f(0, 0, 2\pi) - f(\sqrt{2\pi}, 0, 0) = 2\pi.$$

This model, like all mathematical models, is unrealistic for any number of reasons!

Here are the answers to the easier exercises 3(d) and 6. Please have a go at them if you have time.

(c) # 3(d).

The parabola may be parametrised by $(x, y, z) = c(t) = (t, 0, t^2)$, $-1 \leq t \leq 1$. Thus, the integral is

$$\int_{-1}^{1} (t^2, -t \cdot 0, 1) \cdot (1, 0, 2t) dt = \int_{-1}^{1} (t^2 + 2t) dt = 2/3.$$

(d) # 6.

Here $c'(t) = (1, 2t, 3t^2)$ and $\mathbf{F}(c(t)) = (t^2, 2t, t^2)$. So the desired integral is

$$\int_{0}^{1} (t^2 + 4t^2 + 3t^4) dt = \int_{0}^{1} (5t^2 + 3t^4) dt = 5/3 + 3/5 = 34/15.$$

(3) Let C be the solid region bounded by the cone $(z - 1)^2 = x^2 + y^2$ and the planes $z = 0$ and $z = 1$.

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (x + 2z, y/2, z)$.

(a) Sketch C and $T(C)$.

C is a cone with base the circle $x^2 + y^2 = 1$ and vertex at $(0, 0, 1)$. $T(C)$ is the same cone, but squashed by a factor of $1/2$ in the y–direction and skewed, so that its vertex is now at $(2, 0, 1)$.

(b) Calculate the volume of $T(C)$.

The change of variables formula tells us that the volume of $T(C)$ is

$$\iiint_{T(C)} 1 dV = \iiint_{C} 1 \cdot |J| dV$$

where J is the Jacobian of T. The Jacobian is the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is $1/2$. Therefore, the volume of $T(C)$ is half the volume of C. So we just need to compute the volume of C. This is the volume of a cone of radius 1 and height 1, which is $\pi/3$, as was shown in lectures by using cylindrical coordinates. Therefore, the volume of $T(C)$ is $\pi/6$.

(4) Let $b > a > 0$.

(a) The base of a fence is given by the graph of $y = \log(x)$, $a \leq x \leq b$. The height of the fence is given by $h(x, y) = x^2$. Calculate the area of the fence.

The area of the fence is

$$\int_C h(x,y)ds$$

where C is the section of the graph of $y = \log(x)$ between $x = a$ and $x = b$. C may be paramterized by $c(t) = (t, \log(t))$, $a \leq t \leq b$. Then $c'(t) = (1, 1/t)$. Therefore, the desired integral is

$$\int_a^b (t)^2\sqrt{1+(1/t)^2}dt = \int_a^b t^2\sqrt{1+\frac{1}{t^2}}dt = \int_a^b t\sqrt{t^2+1}dt.$$

This equals $[\frac{1}{3}(t^2+1)^{3/2}]_a^b = \frac{1}{3}((b^2+1)^{3/2} - (a^2+1)^{3/2})$.

(b) Calculate the length of the graph of $y = \log(x)$ between a and b.

This time, we have the same integral but with $h(x,y) = 1$. So we need to compute

$$\int_a^b \|c'(t)\|dt = \int_a^b \sqrt{1+1/t^2}dt = \int_a^b \frac{\sqrt{t^2+1}}{t}dt.$$

Looking at the table of integrals in the textbook, this is number 56 with $a = 1$. So the answer is

$$\left[\sqrt{x^2+1} - \log\left|\frac{1+\sqrt{x^2+1}}{x}\right|\right]_a^b.$$

This could be written as

$$\sqrt{b^2+1} - \sqrt{a^2+1} + \log\left|\frac{b(1+\sqrt{a^2+1})}{a(1+\sqrt{b^2+1})}\right|.$$