MATH 2220 HW9.

Due Wednesday 19 November

(1) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the semicircle $x^2+y^2=1$, $y\geq 0$, oriented anticlockwise, and $\mathbf{F}(x,y)=(-y,x)$.

C may be parametrized as $c(t)=(\cos t,\sin t)$ with $0\leq t\leq\pi$. So the desired integral is

$$\int_0^\pi(-\sin t,\cos t)\cdot(-\sin t,\cos t)dt=\int_0^\pi dt=\pi.$$

(2) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in $\mathbb{R}^3$ given by the parametrization $c(t)=(t,t^2,t^3)$, $0\leq t\leq 1$ and $\mathbf{F}(x,y,z)=(z,1,x^2)$.

This time, we are given the parametrization already. $c'(t)=(1,2t,3t^2)$ so the integral is

$$\int_0^1(t^3,1,t^2)\cdot(1,2t,3t^2)dt=\int_0^1(t^3+2t+3t^4)dt=1/4+1+3/5=37/20.$$

(3) A wire whose shape is given by the curve $(t,\log(t),t^2-1)$, $1\leq t\leq 2$, is made of a material whose density at the point (x,y,z) is $f(x,y,z)=e^{2y}$. Find the mass

$$\int_C f(x,y,z)ds$$

of the wire.

Here $c(t)=(t,\log(t),t^2-1)$ and so $c'(t)=(1,1/t,2t)$. Therefore, $\|c'(t)\|=\sqrt{1+1/t^2+4t^2}=\sqrt{1+t^2+4t^4}/t$. The mass of the wire is given by the integral

$$\int_1^2 f(c(t))\|c'(t)\|dt=\int_1^2 t\sqrt{1+t^2+4t^4}dt.$$

This can be evaluated by using a trick: put $t=\sqrt{u}$. Then $dt=\frac{1}{2}u^{-1/2}du$ and so the integral becomes

$$\frac{1}{2}\int_1^4\sqrt{1+u+4u^2}du.$$

By number 66 in the table of integrals, this equals

$$\left[\frac{8u+1}{16}\sqrt{4u^2+u+1}\right]_1^4+\frac{3}{16}\int_1^4\frac{du}{\sqrt{1+u+4u^2}}.$$

By number 65, the second term may be evaluated:

$$\int_1^4 \frac{du}{\sqrt{1+u+4u^2}} = \left[\frac{1}{2}\log|8u+1+4\sqrt{4u^2+u+1}|\right]_1^4.$$

So the answer may be calculated exactly. Although I may well have made a numerical error, I got

$$\frac{33}{32}\sqrt{69} - \frac{7}{32}\sqrt{6} + \frac{3}{64}\log\left(\frac{33+4\sqrt{69}}{9+4\sqrt{6}}\right).$$

(4) A triangle T with vertices $(0,0,0)$, $(1,1,1)$ and $(-1,-1,1)$ is made of the same material as in the previous question. Find the mass

$$\iint_T f(x,y,z)dS$$

of the triangle. (Hint: an indefinite integral of ue^u is $(u-1)e^u$.)

This is more difficult than the previous problems because we first need to find a parametrization of the surface. Here is one way to do it. First, we find the equation of the plane that contains our triangle. This plane is given by $Ax+By+Cz+D=0$ for some choice of A,B,C,D. Plugging in the three points gives $D=0$ and $A+B+C=0$ and $A+B-C=0$. So we get that the equation of the plane is $x-y=0$. We can therefore parametrize the whole plane by $\Phi(u,v)=(u,u,v)$. But we want just the triangle given in the question, not the whole plane. How to proceed? One way is to take the projection of our triangle onto the plane $y=0$ (the xz–plane). This is the triangle in the uv–plane with vertices $(0,0)$, $(1,1)$ and $(-1,1)$. This may be written as an u–simple region, namely $0 \le v \le 1$, $-v \le u \le v$. Therefore, our whole triangle is described by $\Phi(u,v)=(u,u,v)$ where $0 \le v \le 1$, $-v \le u \le v$.

Now we compute $\Phi_u = (1,1,0)$, $\Phi_v = (0,0,1)$ and $\Phi_u \times \Phi_v = (1,-1,0)$. Thus, $\|\Phi_u \times \Phi_v\| = \sqrt{2}$. Our integral is therefore

$$\int_0^1 \int_{-v}^{v} e^{2u}\sqrt{2}dudv = \int_0^1 \frac{1}{\sqrt{2}}(e^{2v}-e^{-2v})dv = \frac{1}{\sqrt{2}}(e^2/2+e^{-2}/2-1).$$

We didn't need to use the hint! (But we could have, if we had split the region of integration into a pair of v–simple regions instead.)

(5) Let B be the solid region satisfying the equations $z \ge 0$ and $1 \le x^2+y^2+z^2 \le 4$. Let S be the surface of B, oriented outward. Calculate the flux through S of the vector

field $\mathbf{F}(x,y,z) = (z, 0, -1)$. (Hint: S should be split into three parts; a hemisphere of radius 1, a hemisphere of radius 2, and an annulus (ring) in the xy–plane.)

Following the hint, S has three parts. One is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Call it S_2. There is also the hemisphere S_1 given by $x^2 + y^2 + z^2 = 1$, $z \geq 0$. Finally, there is the ring R given by $1 \leq x^2 + y^2 \leq 4$, $z = 0$.

The total flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_R \mathbf{F} \cdot d\mathbf{S}.$$

Each part has to be calculated separately.

To calculate $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, we use spherical coordinates. S_2 is given by the parametrization

$$(x,y,z) = \Phi(\theta,\phi) = (2\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\phi))$$

for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. By a calculation similar to one from the lectures,

$$\Phi_\theta \times \Phi_\phi = -2\sin(\phi)\Phi(\theta,\phi).$$

We want the outward normal, which should have a positive z–component on this part of the surface, so we should choose $-\Phi_\theta \times \Phi_\phi$ as the normal. The flux through S_2 is then

$$\int_0^{2\pi} \int_0^{\pi/2} (2\cos(\phi), 0, -1) \cdot 2\sin(\phi)\Phi(\theta,\phi) d\phi d\theta.$$

This equals

$$\int_0^{2\pi} \int_0^{\pi/2} (4\cos(\phi)\sin^2(\phi)\cos(\theta) - 4\sin(\phi)\cos(\phi)) d\phi d\theta.$$

The first term disappears because $\cos(\theta)$ gets integrated from 0 to 2π. So we are left with $-8\pi \int_0^{\pi/2} \sin(\phi)\cos(\phi) d\phi = -4\pi \int_0^{\pi/2} \sin(2\phi) d\phi = -4\pi$.

The flux through S_1 is calculated similarly. The only differences are that we should take the normal with negative z–component, and that there is no factor of 2 in the parametrization. If you do this calculation, you will end up with π as the answer.

Finally, to do the flux through R, we parametrize R by

$$\Phi(r,\theta) = (r\cos(\theta), r\sin(\theta), 0)$$

with $1 \le r \le 2$ and $0 \le \theta \le 2\pi$. This is just polar coordinates in the plane. We have

$$\Phi_r \times \Phi_\theta = r\mathbf{k}.$$

We want the normal to point straight downwards, so we should take $-r\mathbf{k}$. Now we compute the flux through R:

$$\iint_R \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_1^2 (0, 0, -1) \cdot (0, 0, -r) dr d\theta = \int_0^{2\pi} \int_1^2 r dr d\theta.$$

The value of this integral is 3π.

Thus, the total flux is

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_R \mathbf{F} \cdot d\mathbf{S} = -4\pi + \pi + 3\pi = 0.$$

Note: when we have the divergence theroem, we will be able to do this calculation much more easily. Namely, the flux through S is equal to the integral of the scalar function $\text{div}(\mathbf{F})$ over the region B. But $\text{div}(\mathbf{F}) = 0$ so the flux must be 0.