MATH 2220 PRELIM 1

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) State whether the following are true or false and justify your answer:

(a) The vectors $(5, 7, 1)$ and $(1, 0, -5)$ are orthogonal.

(b) The angle between the vectors $(0, 0, 1)$ and $(5, 7, -1)$ is π.

(c) If L_1 and L_2 are lines in $\mathbb{R}^3$ then there exists a plane containing L_1 and L_2.

(d) If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$ then $\|\mathbf{a} \times \mathbf{b}\| \leq \|\mathbf{a}\|\|\mathbf{b}\|$.

(e) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ then

$$\|(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})\| = \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|.$$

(2) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = \frac{z + 1}{e^x(\cos^2(y) + \frac{1}{2})}$$

(a) State what it means for a function $f : \mathbb{R}^3 \to \mathbb{R}$ to be *continuous*.

(b) Show that f is continuous. You may use the fact that the functions e^x and $\cos(x)$ are continuous functions of one variable.

(c) Either find $\lim_{(x,y,z)\to(0,0,0)} f(x, y, z)$ or show that it does not exist.

(d) Calculate

$$\frac{\partial^5 f}{\partial z \partial z \partial x \partial y \partial y}\Big|_{(0,0,0)}.$$

(3) (a) Find the tangent plane P_a to the surface $z = x^2 - y^3$ at $(1, 1, 0)$.

(b) Find the tangent plane P_b to the surface $yz^2 = 2$ at $(0, 2, 1)$.

(c) Find the line of intersection of the planes P_a and P_b. [TURN OVER]

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function with

$$\frac{\partial f}{\partial x}|_{(0,0)} = 5$$
$$\frac{\partial f}{\partial y}|_{(0,0)} = -3$$

Define $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$g(u, v, w) = f(u + v + w, uvw).$$

(a) Write down a function $h(u, v, w) : \mathbb{R}^3 \to \mathbb{R}^2$ such that $g = f \circ h$.

(b) Write down a formula for $\nabla g(u, v, w)$ in terms of the derivatives of f and h.

(c) Find $\nabla g(0, 0, 0)$.

(d) Find the directional derivative of g at the point $(0, 0, 0)$ in the direction of the vector $(3, 4, 0)$.

(5) The height z of an island above sea level (in meters) is given by

$$z = f(x, y) = \frac{2}{1 + x^2} + \frac{2}{1 + y^2}.$$

(a) Draw the level curve $f(x, y) = k$ for the value $k = 2$.

(b) Calculate ∇f.

(c) A pirate, Captain X, is marooned on the island. He is standing at the point $(-1, 1, 2)$ and wants to descend as quickly as possible. In which direction should he set off?

(d) Captain X has a wooden leg. He will stumble and fall if he has to move in a direction in which the rate of change of f is more than $\frac{1}{2}$ or less than $-\frac{1}{2}$. In which directions could Captain X set off if he wants to descend as quickly as possible but without stumbling?

[END.]