

MATH 2220 PRELIM 1: SOLUTIONS

(1) State whether the following are true or false and justify your answer:

(a) The vectors $(5, 7, 1)$ and $(1, 0, -5)$ are orthogonal.

True, since

$$(5, 7, 1) \cdot (1, 0, -5) = 0.$$

(b) The angle between the vectors $(0, 0, 1)$ and $(5, 7, -1)$ is π .

False. If θ denotes the angle, then the dot product is

$$-1 = \|(0, 0, 1)\| \|(5, 7, -1)\| \cos(\theta)$$

which implies $\cos(\theta) \neq \cos(\pi) = -1$.

(c) If L_1 and L_2 are lines in \mathbb{R}^3 then there exists a plane containing L_1 and L_2 .

False. For example, there is no plane containing both the line $x = y = 0$ and the line $x = 1, z = 0$.

(d) If \mathbf{a} and \mathbf{b} are vectors in \mathbb{R}^3 then $\|\mathbf{a} \times \mathbf{b}\| \leq \|\mathbf{a}\| \|\mathbf{b}\|$.

True. $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta)$ where θ is the smaller angle between \mathbf{a} and \mathbf{b} .

Since $\sin(\theta) \leq 1$, $\|\mathbf{a} \times \mathbf{b}\| \leq \|\mathbf{a}\| \|\mathbf{b}\|$.

(e) If \mathbf{a} , \mathbf{b} and \mathbf{c} are vectors in \mathbb{R}^3 then

$$\|(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})\| = \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|.$$

True. Both sides give twice the area of the triangle in \mathbb{R}^3 whose vertices are \mathbf{a} , \mathbf{b} and \mathbf{c} (cf. Homework 1). The statement could also be proved using algebra.

(2) Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$f(x, y, z) = \frac{z + 1}{e^x (\cos^2(y) + \frac{1}{2})}$$

(a) State what it means for a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ to be *continuous*.

A function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous if for every $\mathbf{x}_0 \in \mathbb{R}^3$, the limit $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x})$ exists and equals $f(\mathbf{x}_0)$. A lot of people got this wrong; do not confuse continuity with differentiability!

(b) Show that f is continuous. You may use the fact that the functions e^x and $\cos(x)$ are continuous functions of one variable.

By using the facts that the sum and product of continuous functions are continuous, this reduces to checking that $1/e^x$ and $1/(\cos^2(y) + \frac{1}{2})$ are continuous. By the rules given in lectures for continuity, these are continuous provided e^x and $\cos^2(y) + \frac{1}{2}$ are never zero, which is the case.

(c) Either find $\lim_{(x,y,z) \rightarrow (0,0,0)} f(x, y, z)$ or show that it does not exist.

Since f is continuous, $\lim_{(x,y,z) \rightarrow (0,0,0)} f(x, y, z) = f(0, 0, 0) = 2/3$.

(d) Calculate

$$\frac{\partial^5 f}{\partial z \partial z \partial x \partial y \partial y} \Big|_{(0,0,0)}.$$

Almost everybody got this right, but please be careful to use ∂ and not d to denote partial differentiation! The partial derivatives may be rearranged, and f_{zz} is already zero, so the answer is 0.

(3) (a) Find the tangent plane P_a to the surface $z = x^2 - y^3$ at $(1, 1, 0)$.

The surface is given by $g(x, y, z) = 0$ where $g(x, y, z) = x^2 - y^3 - z$. A normal vector to the tangent plane is $\nabla(g) = (2x, -3y^2, -1)$. At $(1, 1, 0)$, the equation of the tangent plane is therefore

$$(2, -3, -1) \cdot (x - 1, y - 1, z) = 0$$

or

$$2(x - 1) - 3(y - 1) - z = 0.$$

Note that “ $2(x - 1) - 3(y - 1) - z$ ” is not the equation of anything! Remember that a surface is the set of points satisfying some equation. If there is no $=$ sign in your answer, there is no equation. Some people made this mistake.

(b) Find the tangent plane P_b to the surface $yz^2 = 2$ at $(0, 2, 1)$.

This time, the equation of the surface is $h(x, y, z) = yz^2 - 2 = 0$. Then $\nabla(h) = (0, z^2, 2yz)$ and the tangent plane at $(0, 2, 1)$ is given by the equation

$$(0, 1, 4) \cdot (x - 0, y - 2, z - 1) = 0$$

or $y - 2 + 4(z - 1) = 0$.

(c) Find the line of intersection of the planes P_a and P_b .

The line of intersection is the set of points (x, y, z) which satisfy both equations:

$$2x - 3y - z + 1 = 0$$

$$y + 4z = 6$$

If you have taken a linear algebra course, you should know how to solve this system by row-reduction. In this case, we do not even need to use row-reduction.

We could just write $z = t$ then $y = 6 - 4t$ and $2x - 3(6 - 4t) - t + 1 = 0$ so $2x = 17 - 11t$. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17/2 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -11/2 \\ -4 \\ 1 \end{pmatrix}$$

There are other possible answers and also other methods. For example, the direction of the line can be found by crossing the two normal vectors from parts (a) and (b), and then you only need to find a point which lies on both planes in order to find the equation of the line. Some people did it this way.

(4) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function with

$$\frac{\partial f}{\partial x} \Big|_{(0,0)} = 5$$

$$\frac{\partial f}{\partial y} \Big|_{(0,0)} = -3$$

Define $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$g(u, v, w) = f(u + v + w, uvw).$$

(a) Write down a function $h(u, v, w) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $g = f \circ h$.

$$h(u, v, w) = (u + v + w, uvw).$$

(b) Write down a formula for $\nabla g(u, v, w)$ in terms of the derivatives of f and h .

$$\nabla g(u, v, w) = Dg(u, v, w) = Df(h(u, v, w))Dh(u, v, w) \text{ by the chain rule.}$$

(c) Find $\nabla g(0, 0, 0)$.

$\nabla g(0, 0, 0) = Df(h(0, 0, 0))Dh(0, 0, 0)$. The question tells us that $Df(h(0, 0, 0)) = Df(0, 0) = [5 \quad -3]$. The derivative of h is the matrix of partial derivatives:

$$Dh = \begin{bmatrix} 1 & 1 & 1 \\ vw & uw & uv \end{bmatrix}$$

Thus,

$$Dh(0, 0, 0) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So

$$\nabla g(0, 0, 0) = [5 \quad -3] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = [5 \quad 5 \quad 5]$$

(d) Find the directional derivative of g at the point $(0, 0, 0)$ in the direction of the vector $(3, 4, 0)$.

The vector $(3, 4, 0)$ is not a unit vector, so it should be normalized by dividing by $\sqrt{3^2 + 4^2 + 0^2} = 5$. The directional derivative is then given by

$$\nabla g(0, 0, 0) \cdot \left(\frac{3}{5}, \frac{4}{5}, 0\right) = 3 + 4 = 7.$$

(5) The height z of an island above sea level (in meters) is given by

$$z = f(x, y) = \frac{2}{1+x^2} + \frac{2}{1+y^2}.$$

(a) Draw the level curve $f(x, y) = k$ for the value $k = 2$.

The level curve is the set of (x, y) which satisfy $\frac{2}{1+x^2} + \frac{2}{1+y^2} = 2$, in other words $\frac{1}{1+x^2} + \frac{1}{1+y^2} = 1$. Multiplying through by $(1+x^2)(1+y^2)$, this is the same as $(1+x^2+1+y^2) = (1+x^2)(1+y^2) = 1+x^2+y^2+x^2y^2$, which simplifies to

$x^2y^2 = 1$, i.e. $y = \pm 1/x$. Your picture should therefore consist of the hyperbolas $y = 1/x$ and $y = -1/x$.

(b) Calculate ∇f .

$$\nabla f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right)$$

Calculating the partial derivatives yields $\frac{\partial f}{\partial x} = -4x/(1+x^2)^2$ and $\frac{\partial f}{\partial y} = -4y/(1+y^2)^2$. So

$$\nabla f = \left(\frac{-4x}{(1+x^2)^2} \quad \frac{-4y}{(1+y^2)^2} \right)$$

(c) A pirate, Captain X, is marooned on the island. He is standing at the point $(-1, 1, 2)$ and wants to descend as quickly as possible. In which direction should he set off?

He should set off in the direction in which the rate of change of height is as small as possible. This direction is $-\nabla f(-1, 1)$ (minor notational point: a lot of people wrote $\nabla f(-1, 1, 2)$ which does not make sense since f is a function of only two variables). Substituting into the formula from the previous part yields

$$-\nabla f(-1, 1) = -(1, -1) = (-1, 1).$$

(d) Captain X has a wooden leg. He will stumble and fall if he has to move in a direction in which the rate of change of f is more than $\frac{1}{2}$ or less than $-\frac{1}{2}$. In which directions could Captain X set off if he wants to descend as quickly as possible but without stumbling?

Only a few people got this correct. It is important to read the question carefully (not always easy if you are short of time). Captain X wants to descend as quickly as possible, so he should go in a direction in which the slope of the island (i.e. the rate of change of f) is as negative as possible. The most negative allowed value is $-1/2$. So we need to find a unit vector (v_1, v_2) such that $\nabla f(-1, 1) \cdot (v_1, v_2) = -1/2$. Since $\nabla f(-1, 1) = (1, -1)$, this gives the equation $v_1 - v_2 = -1/2$. The vector (v_1, v_2) is also supposed to be a unit vector, so has to satisfy $v_1^2 + v_2^2 = 1$.

Substituting $v_1 = v_2 - 1/2$ yields the quadratic equation

$$(v_2 - \frac{1}{2})^2 + v_2^2 = 1$$

or $2v_2^2 - v_2 - \frac{3}{4} = 0$. Solving using the quadratic formula yields $v_2 = (1 \pm \sqrt{1+6})/4 = \frac{1 \pm \sqrt{7}}{4}$. For each choice of v_2 , there is one choice of v_1 , namely $v_2 - \frac{1}{2}$. Therefore, there are two possible directions:

$$(\frac{1 + \sqrt{7}}{4} - \frac{1}{2}, \frac{1 + \sqrt{7}}{4})$$

and

$$(\frac{1 - \sqrt{7}}{4} - \frac{1}{2}, \frac{1 - \sqrt{7}}{4})$$