MATH 2220 PRELIM (PRACTICE)

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. **The exam is printed on both sides of the paper.**

Good luck!

(1) Calculate:

(a)

$$\int_0^1 \int_0^x \sin(y) dy dx.$$

(b)

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{3y^2}{x^7+1} dx dy.$$

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x(y + 1).$$

(a) Find all critical points of $f(x, y)$ and determine their nature.

(b) Find the absolute maximum and minimum of $f(x, y)$ on the disc $x^2 + y^2 \leq 3$.

(3) Let Ω be the region enclosed by the plane $2x + 2y + z = 7$ and the paraboloid $z = x^2 + y^2$.

(a) Sketch Ω.

(b) Set up a triple integral for the volume of Ω. Be sure to write all the limits of integration, but do not attempt to evaluate the integral.

(4) A rectangular cuboid has volume $100 cm^3$. Use Lagrange multipliers to find:

(a) The minimum possible surface area of the cuboid.

(b) The minimum possible sum of the lengths of the edges of the cuboid. [TURN OVER]

(5) Let

$$f(x,y) = \frac{x}{\cos^2(y)}$$

(a) Find the second-order Taylor polynomial of f about $(0,0)$. You may omit the remainder term.

(b) Show that the equation

$$f(x,y) + f(x,z) + f(z,x) = 0$$

uniquely determines z as a function of x and y near the point $(0,0,0)$.

(c) Find $\frac{\partial z}{\partial x}|_{(0,0)}$.

[END OF PAPER.]

EXTRA QUESTIONS

Note: some of these are harder than what is likely to be on the exam. Also, see the textbook for more practice problems.

(1) Find the maximum possible volume of a circular cylinder with cross-sectional radius r and surface area S.

(2) (Prelim 2 07) Let $x > 1$ and $y > 1$. Use Lagrange multipliers to find the maximum value of x^y subject to the constraint $xy = 1$.

(3) Solve the previous problem using one-variable calculus.

(4) Find the absolute maximum and minimum of $f(x,y) = 3x+y$ on the disc $x^2+y^2 \leq 1$.

(5) Find numbers A_i such that $4x^2 + 3xy$ equals $A_0 + A_1(x-1) + A_2(y-1) + A_3(x-1)^2 + A_4(x-1)(y-1) + A_5(y-1)^2$.

(6) Use double integration to calculate the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$.

(7) Set up a triple integral for the volume enclosed by the parabolic cylinders $z = x^2$ and $z = 1 - y^2$.

(8) Suppose $y^5 + 2y + 3x = 0$.

(a) Use the implicit function theorem to show that this equation uniquely defines y as a function of x near $(-1,1)$.

(b) Calculate $\frac{dy}{dx}|_{x=-1}$.

(c) Find the second order Taylor polynomial of y near $(-1, 1)$.

(9) Show that $f(x, y) = xy$ has only one critical point and that it is a saddle. Sketch the graph of f. How is f related to the function $g(x, y) = x^2 - y^2$?

(10) Find the absolute maximum and minimum of $x\cos(y)$ on the rectangle $[0, 1] \times [0, 1]$.

(11) A very large tombstone is to be made in the shape of the box $[0, 1] \times [0, 5] \times [0, 10]$ in $\mathbb{R}^3$ (lengths are measured in feet). The tombstone is made of marble whose density at the point (x, y, z) is given by $100 - z^2$ kilograms per cubic foot. Set up a triple integral for the mass of the tombstone and compute it.

(12) Find the volume between the paraboloids $z = 6 - x^2 - y^2$ and $z = x^2 + y^2$ (you may need to use a table of integrals).

(13) Let $a, b, D > 0$. The hull of a ship is to consist of those points in $\mathbb{R}^3$ which lie above the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ in the xy–plane and which lie below the plane $\frac{D}{b}y + D = z$ and below the plane $-\frac{D}{b}y + D = z$.

(a) Make a rough sketch of the hull.

(b) Calculate the volume of the hull (hint: it is easier to calculate half the volume. That is, the volume of the solid that lies above the region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, $y \leq 0$ and below the plane $\frac{D}{b}y + D = z$).

(c) Given the constraint $a + b + D = k$, find the values of a, b and D which maximize the volume of the hull.