MATH 2220 PRELIM 2

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) (a) Find all the critical points of

$$f(x,y) = x^2 - 6xy + 10y^2$$

and determine their nature.

Solution: The gradient $\nabla f = (2x - 6y, -6x + 20y)$. So the equation $\nabla f = 0$ is equivalent to

$$\begin{pmatrix} 2 & -6 \\ -6 & 20 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This has only one solution $(0,0)$, therefore $(0,0)$ is the only critical point. To find the nature of the critical point, observe that the Hessian matrix is

$$H = Hf(0,0) = \begin{pmatrix} 2 & -6 \\ -6 & 20 \end{pmatrix}$$

We have $\det(H) = 2.20 - 36 > 0$ and $\text{trace}(H) = 2 + 20 = 22 > 0$. Therefore, $(0,0)$ is a local minimum.

[Notice that $f(x,y) = (x-3y)^2 + y^2$, and the minimum value is clearly zero when $x - 3y = 0$ and $y = 0$, confirming what we found above.]

(b) Find the second order Taylor polynomial of

$$g(x,y) = x^2 e^x - 6x(e^y - 1) - 20(\cos(y) - 1)$$

about the point $(0,0)$ (you may omit the remainder term).

Solution: The ingredients are obtained by differentiating g.

$$\begin{aligned}
g_x &= 2xe^x + x^2e^x - 6(e^y - 1)\\
g_{xx} &= 2e^x + 2xe^x + 2xe^x + x^2e^x\\
g_y &= -6xe^y + 20\sin(y)\\
g_{yy} &= -6xe^y + 20\cos(y)\\
g_{xy} &= -6e^y
\end{aligned}$$

Evaluating at $(0,0)$ gives

$$\begin{aligned}
g_x(0,0) &= 0\\
g_{xx}(0,0) &= 2\\
g_y(0,0) &= 0\\
g_{yy}(0,0) &= 20\\
g_{xy}(0,0) &= -6
\end{aligned}$$

Also, $g(0,0) = 0$. Therefore, the second order Taylor polynomial is

$$f(h_1,h_2) \approx 0 + 0h_1 + 0h_2 + \frac{1}{2}\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} 2 & -6\\ -6 & 20 \end{bmatrix}\begin{bmatrix} h_1\\ h_2 \end{bmatrix}$$

which equals

$$h_1^2 - 6h_1h_2 + 10h_2^2.$$

(2) Calculate

$$\int_0^1 \int_{\sqrt[3]{y}}^1 e^{x^4}\,dxdy.$$

Solution: The given iterated integral is the double integral of the function $f(x,y) = e^{x^4}$ over the region $0 \le y \le 1$, $\sqrt[3]{y} \le x \le 1$ in the xy–plane. This region may also be written as

$$\begin{aligned}
0 \le x \le 1\\
0 \le y \le x^3
\end{aligned}$$

(You should draw a diagram of the region to see this.) So the integral can be written as

$$\int_0^1 \int_0^{x^3} e^{x^4}\, dy dx.$$

This integral may be evaluated directly. Integrating with respect to y gives

$$\int_0^1 \int_0^{x^3} e^{x^4}\, dy dx = \int_0^1 x^3 e^{x^4}\, dx = \frac{1}{4} e^{x^4}|_0^1 = \frac{1}{4}(e-1).$$

(3) (a) The number of cookies which can be baked from x pounds of flour, y pounds of sugar and z pounds of butter is

$$C(x,y,z) = xyz + 3.$$

Assuming that we can buy a total of six pounds of ingredients, what is the maximum number of cookies which can be produced?

Solution: Lagrange multipliers: we want to optimize the function $C(x,y,z)$ with the constraint $g(x,y,z) = x+y+z = 6$. The equations $\nabla f = \lambda \nabla g$ become

$$yz = \lambda$$

$$xz = \lambda$$

$$xy = \lambda$$

So $xy = yz = xz$. Now *if* $x, y, z \neq 0$ we can cancel and get $x = y = z$. We may assume that $x, y, z \neq 0$ if we are looking for a maximum, by the physical nature of the problem. The constraint $x + y + z = 6$ then yields $x = y = z = 2$ so the maximum is 11 cookies. [Note that a maximum does in fact exist, because the constraint set $x + y + z = 6$, $x, y, z \geq 0$ is closed and bounded. In general, we would not be able to guarantee that a maximum does exist in a problem like this. Nobody noticed this!]

(b) Find the maximum and minimum values of the function $f(x,y) = x^4 + y^4$ on the disc $x^2 + y^2 \leq 2$.

Solution: Looking at the interior first, we look for critical points of f in the region $x^2 + y^2 < 2$. We have $\nabla f = (4x^3, 4y^3)$. The only critical point is $(0,0)$

and it does lie in the open disc. Next, we look at the boundary circle $x^2+y^2 = 2$. There are two possible methods to deal with the boundary.

Method 1: Lagrange multipliers. We wish to optimize x^4+y^4 subject to $x^2+y^2 = 2$. We need to solve the equations

$$\begin{aligned} 4x^3 &= 2\lambda x \\ 4y^3 &= 2\lambda y \\ x^2 + y^2 &= 2 \end{aligned}$$

if $x \neq 0$ *and* $y \neq 0$ we get $2x^2 = 2y^2$ from the first two equations, whence $x = \pm y$ and so $(x, y) = (\pm 1, \pm 1)$. We also have to deal with the case $x = 0$, which yields $y^2 = 2$ so $y = \pm\sqrt{2}$, and $y = 0$ which yields $x = \pm\sqrt{2}$. A lot of people missed these solutions. Thus, we must consider the following points: $(0, 0), (\pm 1, \pm 1), (\pm\sqrt{2}, 0), (0, \pm\sqrt{2})$. Evaluating f at each of these points shows that the minimum value is 0 and the maximum value is $(\pm\sqrt{2})^4 = 4$.

Method 2: Parametrization. We write the boundary curve in terms of a new variable t. Put $x(t) = \sqrt{2}\cos(t)$, $y(t) = \sqrt{2}\sin(t)$, $0 \leq t \leq 2\pi$. Then on the boundary, $g(t) = f(x(t), y(t)) = 4\cos^4(t) + 4\sin^4(t)$. To find the maximum and minimum values of f on the boundary, we must look for critical points of this function.

$$g'(t) = -16\cos^3(t)\sin(t) + 16\sin^3(t)\cos(t) = 0$$

implies

$$\sin(t)\cos(t)(\sin(t) + \cos(t))(\sin(t) - \cos(t)) = 0.$$

The solutions to this are $t = k\pi/4$ with k an integer. We need to check $t = 0$ and $t = 2\pi$, but these are already on our list. The possible points $(x(t), y(t))$ with $t = k\pi/4$ are the same as the points found with Method 1. Then proceed to compare the values of f at these points as above.

(4) Suppose x, y, z, w are related by the following equations.

$$xe^z + 2w + 5y = 0$$

$$w + z + yz^5 + 4w^6 = 0$$

(a) Show that these equations uniquely determine z and w as functions of x and y near the point $(0, 0, 0, 0)$.

Solution: The inverse function theorem tells us to check that the determinant

$$\det \begin{pmatrix} (F_1)_z & (F_1)_w \\ (F_2)_z & (F_2)_w \end{pmatrix}$$

evaluated at $(0, 0, 0, 0)$ is nonzero, where

$$F_1(x, y, z, w) = xe^z + 2w + 5y$$

$$F_2(x, y, z, w) = w + z + yz^5 + 4w^6$$

So we need to look at

$$\begin{pmatrix} xe^z & 2 \\ 1 + 5yz^4 & 1 + 24w^5 \end{pmatrix}$$

Plugging in $(0, 0, 0, 0)$ gives the matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

which has determinant $-2 \neq 0$, as required.

(b) Calculate $\frac{\partial z}{\partial x}|_{(0,0)}$ and $\frac{\partial w}{\partial x}|_{(0,0)}$.

We must differentiate the equations

$$F_1(x, y, z, w) = xe^z + 2w + 5y = 0$$

$$F_2(x, y, z, w) = w + z + yz^5 + 4w^6 = 0$$

with respect to x. We get

$$e^z + xe^z \frac{\partial z}{\partial x} + 2\frac{\partial w}{\partial x} = 0$$

and

$$\frac{\partial w}{\partial x} + \frac{\partial z}{\partial x} + 5yz^4 \frac{\partial z}{\partial x} + 24w^5 \frac{\partial w}{\partial x} = 0.$$

Now we plug in $x = y = 0$ and use the fact that $z(0,0) = w(0,0) = 0$. This gives the equations

$$1 + 2\frac{\partial w}{\partial x}|_{(0,0)} = 0$$
$$\frac{\partial w}{\partial x}|_{(0,0)} + \frac{\partial z}{\partial x}|_{(0,0)} = 0$$

from which we get $\frac{\partial w}{\partial x}|_{(0,0)} = -\frac{1}{2}$ and $\frac{\partial z}{\partial x}|_{(0,0)} = \frac{1}{2}$.

(5) A paperweight P consists of the points in $\mathbb{R}^3$ lying above the triangle with vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ and below the plane $z = 3x + 2y$.

(a) Calculate the volume of the paperweight.

Solution: The volume of the paperweight is the double integral of the function $f(x,y) = 3x + 2y$ over the triangle in the plane with vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$. Notice that the value of $f(x,y)$ is nonnegative for every point in the triangle, and so the region in the plane over which we should integrate is indeed the given triangle and not some smaller region. The triangle may be written $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$ as a y–simple region (draw a picture!). The integral is then

$$\int_0^1 \int_0^{1-x} (3x + 2y) dy dx$$

which equals

$$\int_0^1 (3x(1-x) + (1-x)^2) dx = \int_0^1 (1 + x - 2x^2) dx = 1 + \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.$$

(b) Suppose the paperweight is made of a material whose density at the point (x,y,z) is $f(x,y,z) = 6 - z$. Set up an iterated integral for the mass

$$M = \iiint_P f(x,y,z) dV$$

of the paperweight. Be sure to include all limits of integration, but do not attempt to evaluate the integral.

Solution: As an elementary region, the paperweight may be described by the equations

$$0 \leq x \leq 1$$

$$0 \leq y \leq 1 - x$$

$$0 \leq z \leq 3x + 2y$$

Thus the integral is

$$\int_0^1 \int_0^{1-x} \int_0^{3x+2y} (6 - z) dz dy dx.$$

Alternatively, we may also describe P as

$$0 \leq y \leq 1$$

$$0 \leq x \leq 1 - y$$

$$0 \leq z \leq 3x + 2y$$

and get the integral

$$\int_0^1 \int_0^{1-y} \int_0^{3x+2y} (6 - z) dz dx dy.$$