MATH 2220 PRELIM (PRACTICE)

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. **The exam is printed on both sides of the paper.**

Good luck!

(1) State whether the following are true or false and justify your answer:

(a) The volume of the parallelepiped spanned by the vectors $(0,0,1)$, $(0,1,1)$ and $(-1,-1,-1)$ is 1.

(b) The length of the vector $\mathbf{i} \times \mathbf{j}$ is $\sqrt{2}$.

(c) If L_1 and L_2 are distinct lines through the origin in $\mathbb{R}^3$ then there exists a unique plane containing L_1 and L_2.

(d) If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$ then $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$.

(e) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

(2) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^2 + y^2 - 1.$$

Let S be the surface $f = 0$ in $\mathbb{R}^3$.

Also define $c : \mathbb{R} \to \mathbb{R}^3$ by $c(t) = (\sin(t), \cos(t), t)$.

(a) Show that $c(t)$ lies on the surface S for every $t \in \mathbb{R}$.

(b) Find the tangent vector $c'(t_0)$ to $c(t)$ at an arbitrary $t_0 \in \mathbb{R}$.

(c) Find the tangent plane to S at a point (x_0, y_0, z_0) on S.

(d) Bob says that $c'(\pi)$ must lie in the tangent plane to S at $(0, -1, \pi)$, because a theorem from the lectures says so. Joe disagrees; he points out that the tangent plane to S at $(0, -1, \pi)$ is given by $y = -1$, but $c'(\pi) = (-1, 0, 1)$, which does not lie in the tangent plane. A heated argument erupts. How would you defuse the situation? [TURN OVER]

(3) (a) Find the Hessian matrix of $f(x,y) = x^3 + y^4 + 2xy$ at $(0,0)$.

(b) Find the tangent plane P_b to the surface $z = x^3 + y^4 + 2xy$ at $(1,-1,0)$.

(c) Find the distance from the point $(0,0,0)$ to the plane P_b.

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function with

$$f(1,0) = -2$$
$$\frac{\partial f}{\partial x}|_{(1,0)} = 7$$
$$\frac{\partial f}{\partial y}|_{(1,0)} = 2$$

Define $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$g(x,y,z) = (f(x,y))^2 + 4f(x,y) + 5z.$$

(a) Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(a,b) = a^2 + 4a + 5b$. Find a function $v : \mathbb{R}^3 \to \mathbb{R}^2$ such that $g = u \circ v$.

(b) State the chain rule for the composition $g = u \circ v$.

(c) Find $\nabla g(1,0,0)$.

(d) Find the directional derivative of g at the point $(1,0,0)$ in the direction of the vector $(1,1,1)$.

(5) The height z of a mountain (in feet) is given by

$$z = f(x,y) = 2000(e^{-x^2-y^2}).$$

(a) Draw the level curve $f(x,y) = k$ for the value $k = 1000$.

(b) Calculate ∇f.

(c) Klaus is standing at the point on the mountain with $x = 0$ and $y = 5$. If he starts walking in the direction $(1,5)$, what is the gradient (ie. what is the rate of change of f in the direction $(1,5)$)?

[END OF PAPER.]

EXTRA QUESTIONS

(1) If $\mathbf{a}$ is a vector in $\mathbb{R}^3$, show that there exist vectors $\mathbf{b}$ and $\mathbf{c}$ with $\mathbf{a} = \mathbf{b} \times \mathbf{c}$.

(2) If $\mathbf{a}$ is a vector in $\mathbb{R}^3$ and we know $\mathbf{a} \times \mathbf{i}$ and $\mathbf{a} \times \mathbf{j}$, do we know $\mathbf{a}$?

(3) If $\mathbf{a}$ is a vector in $\mathbb{R}^3$ and we know $\mathbf{a} \cdot \mathbf{i}$ and $\mathbf{a} \cdot \mathbf{j}$, do we know $\mathbf{a}$?

(4) If $\mathbf{a}$ is a vector in $\mathbb{R}^3$ and we know $\mathbf{a} \times \mathbf{i}$ and $\mathbf{a} \times \mathbf{j}$ and $\mathbf{a} \times \mathbf{k}$, do we know $\mathbf{a}$?

(5) If $\mathbf{a}$ is a vector in $\mathbb{R}^3$ and we know $\mathbf{a} \cdot \mathbf{i}$ and $\mathbf{a} \cdot \mathbf{j}$ and $\mathbf{a} \cdot \mathbf{k}$, do we know $\mathbf{a}$?

(6) State the formula for the distance from a given point in $\mathbb{R}^3$ to a given plane.

(7) Find the area of the triangle with vertices $(4, 4, 3)$, $(2, 0, 9)$ and $(3, 1, 5)$.

(8) Find the area of the triangle in $\mathbb{R}^2$ with vertices $(9, 9)$, $(3, 3)$ and $(7, 1)$.

(9) A mathematician tells you that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 1.$$

Explain precisely what is meant by this statement.

(10) Sketch the surface $y = (z - 1)^2$ in $\mathbb{R}^3$.

(11) Find the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at an arbitrary point (x_0, y_0, z_0).

(12) Given two planes

$$A_1x + B_1y + C_1z + D_1 = 0$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

in $\mathbb{R}^3$, let $\mathbf{n_1} = (A_1, B_1, C_1)$ and $\mathbf{n_2} = (A_2, B_2, C_2)$. Show that the two planes intersect if and only if $\mathbf{n_1} \times \mathbf{n_2} \neq \mathbf{0}$.

(13) Either calculate the value of

$$\lim_{(x,y)\to(0,0)} \frac{x^{1/3} + y^{1/3}}{x^{2/3} + y}$$

or show that it does not exist.

(14) Show that the function $f(x, y, z) = \frac{1}{z}(e^x + y + y^2)$ is C^1 where defined, and calculate its derivative at an arbitrary point in its domain.

(15) Explain why the set

$$\{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$$

is open, where $a < b$ and $c < d$. Draw a sketch of the set.

(16) (Final exam 07)

Let $f(x, y)$ be a C^2 function of two variables satisfying $f(3, -2) = 1$, $f_x(3, -2) = 2$, $f_y(3, -2) = 1$, $f_{xx}(3, -2) = 3$, $f_{yy}(3, -2) = 2$, $f_{xy}(3, -2) = 4$. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(u, v) = f(u + 2v, u - 3v)$.

Use the chain rule to compute the derivative matrix Dg at $(1, 1)$. Also, calculate the value of g_{uv} at $(1, 1)$.

(17) You have a nightmare in which a mysterious cloaked figure threatens to destroy the universe unless you can calculate $f_{xyzxyzxyz}$ within the next ten seconds, where

$$f(x, y, z) = \sin(x + 2y + \tan^{-1}(xy^5))e^{400x+(xy+9)^8}(z^2 + 94z - 9).$$

How should you respond?

(18) Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) = g(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^4 + 3xy^2 - y^4$. Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

(19) Use the definition of the derivative to check that the derivative of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x(y + 1)$ at $(0, 0)$ is $\begin{bmatrix} 1 & 0 \end{bmatrix}$.

(20) p.172, # 16.

(21) For more practice problems, look at the Review Exercises on pp. 173-180 of the textbook.