MATH 2220: GUIDE TO INTEGRALS

There are many kinds of integrals in this course. This brief guide is supposed to help you to tell them apart.

(1) Standard integral of a function $f(x)$ of one variable on an interval $[a, b]$.

$$\int_a^b f(x)dx.$$

Represents area under the graph of f between $x = a$ and $x = b$.

(2) Path integral of a scalar function f along a curve C with parametrization $c(t)$, $a \leq t \leq b$ in $\mathbb{R}^3$.

$$\int_C f(x, y, z)ds = \int_a^b f(c(t))\|c'(t)\|dt.$$

Represents mass of a wire with shape C and density $f(x, y, z)$.

Path integral of a scalar function f along a curve C with parametrization $c(t)$, $a \leq t \leq b$ in $\mathbb{R}^2$.

$$\int_C f(x, y)ds = \int_a^b f(c(t))\|c'(t)\|dt.$$

Represents area of a curtain with base C and height $f(x, y)$.

These two integrals don't depend on the choice of parametrization of C. Special case: $f = 1$ gives arc length.

(3) Path (or line) integral of a vector field $\mathbf{F}$ along a curve C with parametrization $c(t)$, $a \leq t \leq b$ in $\mathbb{R}^3$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(c(t)) \cdot c'(t)dt.$$

- Represents work done by $\mathbf{F}$ on a particle moving along C.
- Depends on orientation (direction) of C, but not on the choice of parametrization.
- other notations:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (F_1 dx + F_2 dy + F_3 dz).$$

- If C is a simple closed curve, this integral is sometimes written

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

(4) Double integral of a function $f(x, y)$ over a region $\Omega \subset \mathbb{R}^2$.

$$\iint_\Omega f(x, y) dA.$$

Calculated by writing our region as x–simple or y–simple, also sometimes by conversion to polar coordinates or other change of variable. Special case: $f = 1$ gives area.

Change of variable formula: if $T : D \to T(D)$ is one-to-one then

$$\iint_{T(D)} f(x, y) dA = \iint_D f(T(u, v)) |DT(u, v)| du dv.$$

(5) Integral of a scalar function $f(x, y, z)$ over a parametrized surface S with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D \subset \mathbb{R}^2$.

$$\iint_S f(x, y, z) dS = \iint_D f(\Phi(u, v)) \|\Phi_u \times \Phi_v\| du dv.$$

Represents mass of S if S has density $f(x, y, z)$ at a point (x, y, z). Doesn't depend on choice of orientation. Special case: $f = 1$ give surface area.

(6) Integral of a vector field $\mathbf{F}$ over a parametrized surface S with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D \subset \mathbb{R}^2$.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F}(\Phi(u, v)) \cdot (\Phi_u \times \Phi_v) du dv.$$

where $\mathbf{n}$ is a unit normal in the direction $\Phi_u \times \Phi_v$.

- Depends on a choice of unit normal (orientation). Otherwise independent of the parametrization chosen.
- Represents the flux of the field $\mathbf{F}$ through S.

(7) Triple integral of a scalar function $f(x, y, z)$ over a region $B \subset \mathbb{R}^3$.

$$\iiint_B f(x, y, z) dV.$$

Represents mass of a solid with shape B and density f. Special case: $f = 1$ gives volume. Change of variable formula similar to the two-variable case. Two special changes of variables are cylindrical and spherical coordinates.

Cylindrical:

$$\iiint_B f(x,y,z)dV = \iiint_{B_{cyl}} f(r\cos\theta, r\sin\theta, z) r dr d\theta dz.$$

Spherical:

$$\iiint_B f(x,y,z)dV = \iiint_{B_{spher}} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi.$$

(8) **Fundamental Theorem of calculus.** If F is an antiderivative of f then

$$\int_a^b f(x)dx = F(b) - F(a).$$

(9) **Integral of a conservative field.** If f is a differentiable function and C is a curve with parametrization $c(t)$, $t \in [a,b]$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(c(b)) - f(c(a)).$$

(10) **Green's Theorem.** If Ω is any reasonable closed and bounded region in $\mathbb{R}^2$ and $\partial\Omega$ is the boundary curve of Ω with the "anticlockwise" orientation and $\mathbf{F} = (P, Q)$ is a C^1 vector field on Ω then

$$\int_{\partial\Omega} \mathbf{F} \cdot dr = \iint_\Omega (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA.$$

(11) **Stokes' Theorem.** If S is an oriented surface in $\mathbb{R}^3$ and ∂S is the boundary of S (a collection of curves) with the induced orientation and $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^3$ then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

(12) **Divergence Theorem.** If B is a solid region in $\mathbb{R}^3$ and ∂B is the boundary surface of B and $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^3$ then

$$\iint_{\partial B} \mathbf{F} \cdot d\mathbf{S} = \iiint_B \text{div}(\mathbf{F}) dV.$$

Note: the statements of these theorems have deliberately been left a little bit vague, but they apply in all the situations with which we are familiar.