MATH 2310 FINAL EXAM

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. No calculators or notes are allowed. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) (a) State whether each of the following is true or false, giving a brief reason for your answer.

(i) There exists a linear system $A\mathbf{x} = \mathbf{b}$ with exactly three solutions.

(ii) The inverse of the matrix $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$.

(iii) If A is any matrix, then the matrix $A^T A$ is symmetric.

(b) Let V be the vector space of all $n \times n$ matrices. Which of the following is a subspace of V?

(i) The set of all symmetric $n \times n$ matrices.

(ii) The set of all invertible $n \times n$ matrices.

(2) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 1 \\ 2 & 4 & 6 & 11 & 2 \end{bmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Determine the rank of A.

(c) Are the columns of A linearly independent? Explain.

(d) Are the rows of A linearly independent? Explain.

(3) Let W be the subspace of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} x \\ x \end{bmatrix}$ for $x \in \mathbb{R}$.

(a) Find the orthogonal complement $W^\perp$.

(b) Write the vector $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ as

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

with $\mathbf{u}_1 \in W$ and $\mathbf{u}_2 \in W^\perp$.   [TURN OVER.]

(4) Let

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find a nonzero eigenvector for each eigenvalue.

(c) Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$.

(5) Let V be the vector space of polynomials $a + bt$ of degree ≤ 1 and $L: V \to V$ the linear operator defined by

$$L(f) = \frac{df}{dt}.$$

(a) Find the matrix A of L with respect to the ordered basis $B = \{2t, t - 1\}$ of V.

(b) Is the matrix A invertible? Explain.

(6) Consider the following vectors in $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ which is a basis of $\mathbb{R}^3$.

(7) The population of deer in a forest is described by the following model. Let y_k be the number of juvenile deer in year k and let a_k be the number of adult deer in year k. Then

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix}$$

(a) A *nonzero* vector $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$ is called a *steady-state vector* if

$$\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$$

Does the model have a steady-state vector? If so, find one.

(b) Suppose deer are culled at a rate of 1 juvenile and 2 adults per year. Now the model becomes

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

By solving an appropriate linear system, find a vector $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$ such that $\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$. Also, show that there is no other vector with this property.

(c) Now suppose that we want to cull c_1 juveniles and c_2 adults per year. The model becomes

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} + \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

Show that there is a unique vector $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$ such that $\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$.

(d) A spokesperson for the National Venison Union claims that the answer to part (c) shows that we can cull any number of deer that we want, and the population can still be sustainably managed. Explain why this reasoning is flawed.

(8) Let V be a vector space, let $L: V \to V$ be a linear transformation, and let $\mathbf{x}, \mathbf{y}$ be nonzero vectors in V such that

$$L(\mathbf{x}) = a\mathbf{x}, \ L(\mathbf{y}) = b\mathbf{y},$$

where a, b are real numbers.

(a) State what it means for $L : V \to V$ to be a linear transformation.

(b) State what it means for vectors $\mathbf{x}$, $\mathbf{y}$ to be linearly independent.

(c) Using the definition from part (b), prove that if $a \neq b$, then $\mathbf{x}$, $\mathbf{y}$ are linearly independent.

[END.]