MATH 2310 FINAL EXAM (PRACTICE)

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. No calculators or notes are allowed. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) (a) State whether each of the following is true or false, giving a brief reason for your answer.

(i) There exists an invertible 2×3 matrix.

False: only square matrices can be invertible.

(ii) If $\mathbf{x}$ is a vector, then $\mathbf{x} \cdot \mathbf{x} \geq 0$.

True. $\mathbf{x} \cdot \mathbf{x} = ||\mathbf{x}||^2$.

(iii) If A is any matrix, then the matrix $A + A^T - 3I$ is symmetric.

True. Using properties of the transpose:

$$(A + A^T - 3I)^T = A^T + (A^T)^T - 3I^T = A^T + A - 3I = A + A^T - 3I.$$

(b) Let V be the vector space of all polynomials of degree $\leq n$. Which of the following is a subspace of V?

(i) The set of all polynomials $p(x)$ such that $p'(x) = 0$.

This is a subspace - it's just the constant polynomials, ie. the scalars, which are closed under addition and scalar multiplication, so form a subspace.

(ii) The set of all polynomials $p(x)$ such that $p(-1) = 0$.

A polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfies $p(-1) = 0$ if and only if $a_0 - a_1 + a_2 - \cdots + (-1)^n a_n = 0$. This can be written

$$\sum_{i=0}^{n} (-1)^i a_i = 0.$$

Now if $\sum_{i=0}^{n} (-1)^i a_i = 0$ and $c \in \mathbb{R}$, then $\sum_{i=0}^{n} (-1)^i c a_i = 0$, so the set of polynomials satisfying $\sum_{i=0}^{n} (-1)^i a_i = 0$ is closed under scalar multiplication. Also, if $\sum_{i=0}^{n} (-1)^i a_i = 0$ and $\sum_{i=0}^{n} (-1)^i b_i = 0$ then $\sum_{i=0}^{n} (-1)^i (a_i + b_i) = 0$, so if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$ are

polynomials with $p(-1) = q(-1) = 0$, then also $(p + q)(-1) = 0$. Therefore, the set of polynomials satisfying $p(-1) = 0$ is closed under addition. We conclude that it is indeed a subspace.

(2) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(a) Find all solutions to the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [1, 1, 1]^T$.

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & 6 & 1 \end{bmatrix}$$

We reduce this to reduced row echelon form. First, do $r_2 \rightarrow r_2 - 3r_1$ and $r_3 \rightarrow r_3 - 5r_1$ to obtain

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \end{bmatrix}$$

Then do $r_3 \rightarrow r_3 - 2r_2$ followed by $r_2 \rightarrow r_2/(-2)$ to obtain

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally, doing $r_1 \rightarrow r_1 - 2r_2$ gives the rref

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions are thus $x_1 = -1$, $x_2 = 1$. We conclude that $A\mathbf{x} = \mathbf{b}$ has the unique solution $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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(b) Calculate the rank of A.

From the rref calculated above, we see that the row space of A has dimension 2 (it is spanned by $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$), and so $\operatorname{rank}(A) = 2$.

(3) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, define
$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + 2 x_2 y_2.$$

(a) Show that $(-, -)$ is an inner product on $\mathbb{R}^2$.

We need to check the five axioms of an inner product. Namely,

- $(\mathbf{u}, \mathbf{u}) \geq 0$.
- $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
- $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$.
- $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.
- $(c\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v})$.

for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and all scalars c.

We have $(\mathbf{u}, \mathbf{u}) = u_1^2 + 2u_2^2 \geq 0$, and if this is zero, then $u_1 = u_2 = 0$, so the first two axioms hold. Next, $(\mathbf{v}, \mathbf{u}) = v_1 u_1 + 2 v_2 u_2 = u_1 v_1 + 2 u_2 v_2 = (\mathbf{u}, \mathbf{v})$ for any $\mathbf{u}, \mathbf{v}$. Next, $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (u_1 + v_1) w_1 + 2(u_2 + v_2) w_2 = u_1 w_1 + 2 u_2 w_2 + v_1 w_1 + 2 v_2 w_2 = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$. Finally, $(c\mathbf{u}, \mathbf{v}) = c u_1 v_1 + 2 c u_2 v_2 = c(u_1 v_1 + 2 u_2 v_2) = c(\mathbf{u}, \mathbf{v})$.

(b) Find all vectors which are orthogonal to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under the inner product $(-, -)$.

For $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to be orthogonal to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means that $(\mathbf{u}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = u_1 + 2u_2 = 0$. Thus, the set of all such $\mathbf{u}$ is the set of solutions of the system $u_1 + 2u_2 = 0$. This is $u_1 = -2u_2$, u_2 free, which we may write as

$$\operatorname{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$$

(c) Find two vectors $\mathbf{x}, \mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) = 0$ but $\mathbf{x} \cdot \mathbf{y} \neq 0$.

From the previous part $(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 0$, but we see that $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 + 1 = -1 \neq 0$.

[TURN OVER.]

(4) Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A.

We need to find the characteristic polynomial first. We have

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Via a Laplace expansion, the determinant of this is

$$\det(A - \lambda I) = (2-\lambda)^2(1-\lambda).$$

The eigenvalues are the solutions of

$$(2-\lambda)^2(1-\lambda) = 0.$$

Thus, the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$.

(b) Find the eigenspaces of A.

For the eigenvalue $\lambda_1 = 1$, we have

$$A - \lambda_1 I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we see that the solutions to $(A - I)\mathbf{x} = \mathbf{0}$ are $x_1 + x_3 = 0$, $x_2 = 0$, x_3 free. So the $\lambda = 1$ eigenspace is

$$\text{span}\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$$

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For the eigenvalue $\lambda_2 = 2$, we have

$$A - \lambda_2 I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and so the corresponding eigenspace is

$$\mathrm{Nul}(A - 2I) = \mathrm{span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$$

(c) Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$.

According to our recipe, we may put

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are other possibilities, depending on which eigenvectors we pick from each eigenspace.

(5) Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Show that S is a basis of $\mathbb{R}^2$.

The columns of a matrix form a basis of $\mathbb{R}^n$ if and only if the matrix is invertible (this is part of the invertible matrix theorem). Thus, we need only check that the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is invertible. Its determinant is $1^2 - 2^2 = -3 \neq 0$, so it is invertible. Thus, S is a basis.

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(b) Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right]$. Find the matrix of the linear transformation $L(\mathbf{x}) = A\mathbf{x}$ relative to the basis S.

The matrix we seek is

$$\left[[L(\mathbf{v}_1)]_S \quad [L(\mathbf{v}_2)]_S \right] = \left[[A\mathbf{v}_1]_S \quad [A\mathbf{v}_2]_S \right] = \left[\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]_S \quad \left[\begin{smallmatrix}2\\6\end{smallmatrix}\right]_S \right]$$

To find the columns of this, we express the vector $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ as a linear combination of the vectors in S. That is, we find c_1, c_2 so that

$$\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + c_2\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right]\left[\begin{smallmatrix}c_1\\c_2\end{smallmatrix}\right].$$

We solve this system to obtain

$$\left[\begin{smallmatrix}c_1\\c_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] = (-1/3)\left[\begin{smallmatrix}1&-2\\-2&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}5/3\\-1/3\end{smallmatrix}\right]$$

Thus,

$$\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]_S = \left[\begin{smallmatrix}5/3\\-1/3\end{smallmatrix}\right]$$

and we also have

$$\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right]_S = 2\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]_S = 2\left[\begin{smallmatrix}5/3\\-1/3\end{smallmatrix}\right] = \left[\begin{smallmatrix}10/3\\-2/3\end{smallmatrix}\right]$$

Thus,

$$\left[[L(\mathbf{v}_1)]_S \quad [L(\mathbf{v}_2)]_S \right] = \begin{bmatrix} 5/3 & 10/3 \\ -1/3 & -2/3 \end{bmatrix}$$

(6) Consider the following matrix

$$A = \begin{bmatrix} 6 & 7 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 5 & 1 & 1 & 1 \\ 7 & 2 & 3 & 4 \end{bmatrix}$$

(a) Compute $\det(A)$.

A Laplace expansion along the first row gives

$$\det(A) = 6\begin{vmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} - 7\begin{vmatrix} 4 & 0 & 0 \\ 5 & 1 & 1 \\ 7 & 3 & 4 \end{vmatrix} = 18\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 28\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = -10\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = -10$$

(b) Are the columns of A linearly independent? Explain.

By the invertible matrix theorem, $\det(A) \neq 0$ implies A is invertible implies the columns of A are linearly independent, so yes.

(c) Show that -1 is an eigenvalue of A.

-1 is an eigenvalue if and only if $A - (-1)I = A + I$ is a singular matrix. But

$$\det(A + I) = \det \begin{bmatrix} 7 & 7 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 5 & 1 & 2 & 1 \\ 7 & 2 & 3 & 5 \end{bmatrix} = 28 \det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 1 & 2 & 1 \\ 7 & 2 & 3 & 5 \end{bmatrix}$$

This is the determinant of a matrix with two rows the same, so it is zero. Thus, $A + I$ is singular, and so -1 is an eigenvalue.

(d) Find a vector $\mathbf{v}$ with $A\mathbf{v} + \mathbf{v} = \mathbf{0}$.

Oops! This should say a *nonzero* $\mathbf{v}$. As the question is stated, you can save yourself a lot of work by taking $\mathbf{v} = \mathbf{0}$.

To say $A\mathbf{v} + \mathbf{v} = \mathbf{0}$ is the same as $A\mathbf{v} = -\mathbf{v}$, so we seek an eigenvector $\mathbf{v}$ of A with the eigenvalue -1. Thus, we need to find a nonzero solution of the system

$$(A + I)\mathbf{x} = \begin{bmatrix} 7 & 7 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 5 & 1 & 2 & 1 \\ 7 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Elementary row operations (which I omit, although in an exam you should show all your working) reduce this to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 15/7 & 0 & 1 & 0 \\ -2/7 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain x_1 free, $x_2 + x_1 = 0$, $x_3 + (15/7)x_1 = 0$, $x_4 - (2/7)x_1 = 0$, so taking $x_1 = 7$, one such $\mathbf{v}$ is

$$\mathbf{v} = \begin{bmatrix} 7 \\ -7 \\ -15 \\ 2 \end{bmatrix}$$

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(7) The migration of people between the town and the countryside is described by the following model. Let t_k be the number of people in the town in year k and let c_k be the number of people in the countryside in year k. Then

$$\begin{bmatrix} t_{k+1} \\ c_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} t_k \\ c_k \end{bmatrix}$$

(a) Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$$

(Hint: you may find it easier to find the eigenvalues of $10A$ first. Note that $14^2 = 196$.)

Using the hint,

$$10A = \begin{bmatrix} 9 & 5 \\ 1 & 5 \end{bmatrix}$$

The characteristic polynomial of this is

$$\det(10A - \lambda I) = \begin{bmatrix} 9 - \lambda & 5 \\ 1 & 5 - \lambda \end{bmatrix} = \lambda^2 - 14\lambda + 40 = (\lambda - 10)(\lambda - 4)$$

Thus, the eigenvalues of $10A$ are 10 and 4. Note that if $10A\mathbf{v} = \lambda\mathbf{v}$, then $A\mathbf{v} = (\lambda/10)\mathbf{v}$. Thus, the eigenvalues of A are the eigenvalues of $10A$ divided by 10, that is, 1 and 0.4.

(b) Your friend Joe claims that whatever vector $\begin{bmatrix} t_0 \\ c_0 \end{bmatrix}$ you start with, in the end the vectors $\begin{bmatrix} t_k \\ c_k \end{bmatrix}$ will tend towards a multiple of the vector $\mathbf{z} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. That is, in the long run the population of the town will be five times that of the countryside. Is Joe right? Explain your answer.

Joe is right. Briefly, if you compute the eigenvectors of A, you will find that they are $\mathbf{z} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ (with eigenvalue 1) and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (with eigenvalue 0.4). Every vector can be written as a linear combination of these two vectors, because they are linearly independent and therefore form a basis of $\mathbb{R}^2$. Thus, if you start with an arbitrary vector $\mathbf{v} = a\mathbf{z} + b\mathbf{y}$, you will obtain

$$A^k\mathbf{v} = a\mathbf{z} + b(0.4)^k\mathbf{y}.$$

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As $k \to \infty$, $(0.4)^k$ approaches zero, and thus $A^k\mathbf{v}$ approaches $a\mathbf{z}$, a multiple of $\mathbf{z}$ as claimed.

(8) (a) Give the definitions of the following concepts:

(i) What is meant by a *linear system*?

A linear system is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where the a_{ij}, b_i are real numbers.

(ii) What is meant by the *nullspace* of a matrix?

The nullspace of an $m \times n$ matrix A is the set of vectors $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{0}$.

(iii) What is meant by the *range* of a linear transformation?

The range of a linear transformation $L : V \to W$ is the set of all vectors in W of the form $L(\mathbf{x})$ for $\mathbf{x} \in V$. More formally,

$$\text{range}(L) = \{L(\mathbf{x}) : \mathbf{x} \in V\}.$$

(iv) What is meant by the *kernel* of a linear transformation?

The kernel of a linear transformation $L : V \to W$ is the set of all vectors $\mathbf{x}$ in V such that $L(\mathbf{x}) = \mathbf{0}_W$.

(b) Alice and Bob both solve the following system.

$$3x_1 + 2x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 - x_3 - x_4 = 0$$

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Alice obtains the solution $x_1 = -3t - 3u, x_2 = 4t + 4u, x_3 = t, x_4 = u$, with t, u free. Bob obtains the solution $x_1 = 3t, x_2 = -4t, x_3 = u, x_4 = -t - u$, with t, u free. Who is right?

You can check that both solutions are correct! Alice has written the nullspace of the matrix

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

as

$$\text{span}\left\{ \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Bob has written the same space as

$$\text{span}\left\{ \begin{bmatrix} 3 \\ -4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Alice and Bob have simply found two different bases for the same vector space.

[END.]