MATH 2310 FINAL EXAM (PRACTICE)

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. No calculators or notes are allowed. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** *Good luck!*

(1) (a) State whether each of the following is true or false, giving a brief reason for your answer.

- (i) There exists an invertible 2×3 matrix.
- (ii) If $\mathbf{x}$ is a vector, then $\mathbf{x} \cdot \mathbf{x} \geq 0$.
- (iii) If A is any matrix, then the matrix $A + A^T - 3I$ is symmetric.

(b) Let V be the vector space of all polynomials of degree $\leq n$. Which of the following is a subspace of V?

- (i) The set of all polynomials $p(x)$ such that $p'(x) = 0$.
- (ii) The set of all polynomials $p(x)$ such that $p(-1) = 0$.

(2) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(a) Find all solutions to the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [1, 1, 1]^T$.

(b) Calculate the rank of A.

(3) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, define

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + 2x_2 y_2.$$

(a) Show that $(-,-)$ is an inner product on $\mathbb{R}^2$.

(b) Find all vectors which are orthogonal to the vector $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ under the inner product $(-,-)$.

(c) Find two vectors $\mathbf{x}, \mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) = 0$ but $\mathbf{x} \cdot \mathbf{y} \neq 0$.

[TURN OVER.]

(4) Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find the eigenspaces of A.

(c) Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$.

(5) Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\mathbf{v}_2 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$.

(a) Show that S is a basis of $\mathbb{R}^2$.

(b) Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right]$. Find the matrix of the linear transformation $L(\mathbf{x}) = A\mathbf{x}$ relative to the basis S.

(6) Consider the following matrix

$$A = \begin{bmatrix} 6 & 7 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 5 & 1 & 1 & 1 \\ 7 & 2 & 3 & 4 \end{bmatrix}$$

(a) Compute $\det(A)$.

(b) Are the columns of A linearly independent? Explain.

(c) Show that -1 is an eigenvalue of A.

(d) Find a vector $\mathbf{v}$ with $A\mathbf{v} + \mathbf{v} = \mathbf{0}$.

(7) The migration of people between the town and the countryside is described by the following model. Let t_k be the number of people in the town in year k and let c_k be the number of people in the countryside in year k. Then

$$\begin{bmatrix} t_{k+1} \\ c_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} t_k \\ c_k \end{bmatrix}$$

(a) Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$$

(Hint: you may find it easier to find the eigenvalues of $10A$ first. Note that $14^2 = 196$.)

(b) Your friend Joe claims that whatever vector $\left[\begin{smallmatrix} t_0 \\ c_0 \end{smallmatrix}\right]$ you start with, in the end the vectors $\left[\begin{smallmatrix} t_k \\ c_k \end{smallmatrix}\right]$ will tend towards a multiple of the vector $\mathbf{z} = \left[\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right]$. That is, in the long run the population of the town will be five times that of the countryside. Is Joe right? Explain your answer.

(8) (a) Give the definitions of the following concepts:

(i) What is meant by a *linear system*?

(ii) What is meant by the *nullspace* of a matrix?

(iii) What is meant by the *range* of a linear transformation?

(iv) What is meant by the *kernel* of a linear transformation?

(b) Alice and Bob both solve the following system.

$$3x_1 + 2x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 - x_3 - x_4 = 0$$

Alice obtains the solution $x_1 = -3t - 3u, x_2 = 4t + 4u, x_3 = t, x_4 = u$, with t, u free. Bob obtains the solution $x_1 = 3t, x_2 = -4t, x_3 = u, x_4 = -t - u$, with t, u free. Who is right?

[END.]