

## MATH 2310 FINAL EXAM

You have 2 hours 30 minutes to complete this exam. The exam starts at 7:00pm. Each question is worth 20 marks. There are 8 questions in total. No calculators or notes are allowed. You are free to use results from the lectures, but you should clearly state any theorems you use. **The exam is printed on both sides of the paper.** Good luck!

(1) (a) State whether each of the following is true or false, giving a brief reason for your answer.

(i) There exists a linear system  $A\mathbf{x} = \mathbf{b}$  with exactly three solutions.

*False. A linear system has either 0, 1 or infinitely many solutions.*

(ii) The inverse of the matrix  $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$  is  $\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$ .

*False.  $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \neq I$ .*

(iii) If  $A$  is any matrix, then the matrix  $A^T A$  is symmetric.

*True.  $(A^T A)^T = A^T (A^T)^T = A^T A$ .*

(b) Let  $V$  be the vector space of all  $n \times n$  matrices. Which of the following is a subspace of  $V$ ?

(i) The set of all symmetric  $n \times n$  matrices.

*This is a subspace. If  $A$  and  $B$  are symmetric, then  $(A+B)^T = A^T + B^T = A+B$ , so the sum of symmetric matrices is symmetric. If  $A$  is symmetric and  $c$  is a scalar, then  $(cA)^T = cA^T$ , so the set of symmetric matrices is closed under scalar multiplication.*

(ii) The set of all invertible  $n \times n$  matrices.

*This is not a subspace, since the sum of invertible matrices may not be invertible. For example,  $I$  and  $-I$  are invertible, but  $I + (-I) = 0$  is not.*

(2) Let  $A$  be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 1 \\ 2 & 4 & 6 & 11 & 2 \end{bmatrix}$$

(a) Find a basis for the nullspace of  $A$ .

We solve the system  $A\mathbf{x} = \mathbf{0}$ . The first step is to row-reduce the matrix  $A$ . By subtracting  $2\text{row}_1$  from  $\text{row}_2$ , we see that  $A$  is row-equivalent to

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we subtract  $5\text{row}_2$  from  $\text{row}_1$  to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We obtain the system

$$x_1 + 2x_2 + 3x_3 + x_5 = 0$$

$$x_4 = 0$$

and  $x_2, x_3, x_5$  free. Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 - x_5 \\ x_2 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And a basis for the nullspace of  $A$  is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) Determine the rank of  $A$ .

By the rank-nullity theorem,  $\text{rank}(A) + \dim \text{Nul}(A) = 5$ , so  $\text{rank}(A) = 2$ .

(c) Are the columns of  $A$  linearly independent? Explain.

No. The rank of  $A$  is 2; this is also the dimension of the column space. The 5 columns of  $A$  cannot form a linearly independent set in a two-dimensional space.

(d) Are the rows of  $A$  linearly independent? Explain.

Yes. The rank of  $A$  is 2, so the dimension of the row space of  $A$  is 2. The rows of  $A$  therefore span a space of dimension 2. Since there are exactly two rows, they must be linearly independent.

(3) Let  $W$  be the subspace of  $\mathbb{R}^2$  consisting of the vectors  $\begin{bmatrix} x \\ x \end{bmatrix}$  for  $x \in \mathbb{R}$ .

(a) Find the orthogonal complement  $W^\perp$ .

A basis for  $W$  is given by the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Therefore,  $W^\perp$  is the nullspace of the matrix  $\begin{bmatrix} 1 & 1 \end{bmatrix}$ . This is the set of  $\mathbf{x}$  such that  $x_1 + x_2 = 0$ ,  $x_2$  free. This may be written as

$$W^\perp = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

(b) Write the vector  $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$  as

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

with  $\mathbf{u}_1 \in W$  and  $\mathbf{u}_2 \in W^\perp$ .

We need to find scalars  $c_1$  and  $c_2$  such that  $\begin{bmatrix} 4 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ . That is, we need to solve the system

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This system has a unique solution given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = (1/2) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 1/2 \end{bmatrix}$$

Thus, we have

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

with  $\mathbf{u}_1 = \begin{bmatrix} 9/2 \\ 9/2 \end{bmatrix}$  and  $\mathbf{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

(4) Let

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

(a) Find the eigenvalues of  $A$ .

*To find the eigenvalues, we solve the characteristic equation*

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = 25 - 10\lambda + \lambda^2 - 4 = \lambda^2 - 10\lambda + 21 = 0$$

*which gives  $\lambda = 3, 7$ .*

(b) Find a nonzero eigenvector for each eigenvalue.

*For  $\lambda = 3$ , we find an eigenvector by finding a nonzero solution of  $(A - 3I)\mathbf{x} = \mathbf{0}$ .*

*We must therefore find the nullspace of*

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

*A calculation shows that this is the span of the vector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , so a nonzero eigenvector is  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .*

*For  $\lambda = 7$ , we find an eigenvector by finding a nonzero solution of  $(A - 7I)\mathbf{x} = \mathbf{0}$ .*

*We must therefore find the nullspace of*

$$A - 7I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

*A calculation shows that this is the span of the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , so a nonzero eigenvector is  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .*

(c) Find a diagonal matrix  $D$  and an invertible matrix  $P$  with  $A = PDP^{-1}$ .

*According to our recipe, we may take*

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

*and*

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

*There are other possibilities.*

- (5) Let  $V$  be the vector space of polynomials  $a + bt$  of degree  $\leq 1$  and  $L: V \rightarrow V$  the linear operator defined by

$$L(f) = \frac{df}{dt}.$$

- (a) Find the matrix  $A$  of  $L$  with respect to the ordered basis  $B = \{2t, t - 1\}$  of  $V$ .

Write the basis vectors as  $\mathbf{b}_1 = 2t$  and  $\mathbf{b}_2 = t - 1$ . Then the desired matrix is

$$A = \begin{bmatrix} [L(\mathbf{b}_1)]_B & [L(\mathbf{b}_2)]_B \end{bmatrix} = \begin{bmatrix} [2 + 0t]_B & [1 + 0t]_B \end{bmatrix}.$$

To calculate  $[2 + 0t]_B$ , we need to find the scalars  $c_1, c_2$  with

$$2 + 0t = c_1(2t) + c_2(t - 1)$$

Equating coefficients of  $t$  and  $1$ , we get  $2c_1 + c_2 = 0$  and  $-c_2 = 2$ . This is a linear system whose solution we can write down directly:  $c_2 = -2$  and  $c_1 = -c_2/2 = 1$ .

Therefore,

$$[2 + 0t]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Then we also have

$$[1 + 0t]_B = (1/2)[2 + 0t]_B = (1/2) \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

Thus we have

$$A = \begin{bmatrix} 1 & 1/2 \\ -2 & -1 \end{bmatrix}.$$

- (b) Is the matrix  $A$  invertible? Explain.

No. For example,  $\det(A) = 1(-1) - (-2)(1/2) = 0$ .

- (6) Consider the following vectors in  $\mathbb{R}^3$ .

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Find a subset of  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  which is a basis of  $\mathbb{R}^3$ .

To solve this problem, we can put the vectors into a matrix and reduce it to rref. If the matrix has full rank, then the columns which become pivot columns will then form a basis of  $\mathbb{R}^3$ . Accordingly, we row reduce the matrix

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

First, we do row 3 minus row 1 to obtain

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

Next, we can subtract row 2 from row 1 and add row 2 to row 3 to obtain

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To get the rref, we can divide row 3 by 2 and subtract row 3 from row 1, leaving:

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot columns are the first, second and fourth columns. Thus, a basis is  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ . There are other possible answers.

- (7) The population of deer in a forest is described by the following model. Let  $y_k$  be the number of juvenile deer in year  $k$  and let  $a_k$  be the number of adult deer in year  $k$ . Then

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix}$$

- (a) A nonzero vector  $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$  is called a *steady-state vector* if

$$\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$$

Does the model have a steady-state vector? If so, find one.

*A steady-state vector must satisfy*

$$\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix}$$

*and therefore*

$$\left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

*But the coefficient matrix*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 0 \end{bmatrix}$$

*is invertible, and thus this system has no nonzero solution.*

- (b) Suppose deer are culled at a rate of 1 juvenile and 2 adults per year. Now the model becomes

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

By solving an appropriate linear system, find a vector  $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$  such that  $\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$ . Also, show that there is no other vector with this property.

*As in the previous problem, we need to solve the system*

$$\begin{bmatrix} 1 & -3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

*Since the coefficient matrix is invertible, this has the unique solution*

$$\begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

*There is no other vector with this property because the solution is unique (the coefficient matrix is invertible).*

- (c) Now suppose that we want to cull  $c_1$  juveniles and  $c_2$  adults per year. The model becomes

$$\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} + \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

Show that there is a unique vector  $\begin{bmatrix} y_k \\ a_k \end{bmatrix}$  such that  $\begin{bmatrix} y_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ a_k \end{bmatrix}$ .

Again, we obtain the system

$$\begin{bmatrix} 1 & -3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_k \\ a_k \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

which has a unique solution because the matrix

$$\begin{bmatrix} 1 & -3 \\ -1 & 0 \end{bmatrix}$$

is invertible.

- (d) A spokesperson for the National Venison Union claims that the answer to part (c) shows that we can cull any number of deer that we want, and the population can still be sustainably managed. Explain why this reasoning is flawed.

*Although there is always a steady-state vector, the population of deer has to be large enough in the first place to allow the amount of culling that we want to do. Another problem is that the entries in the steady-state vector may not be whole numbers, but in the real world we cannot have fractions of a deer, so the model will not work properly.*

- (8) Let  $V$  be a vector space, let  $L: V \rightarrow V$  be a linear transformation, and let  $\mathbf{x}, \mathbf{y}$  be nonzero vectors in  $V$  such that

$$L(\mathbf{x}) = a\mathbf{x}, \quad L(\mathbf{y}) = b\mathbf{y},$$

where  $a, b$  are real numbers.

- (a) State what it means for  $L: V \rightarrow V$  to be a linear transformation.

*See Definition 6.1 in the textbook.*

- (b) State what it means for vectors  $\mathbf{x}, \mathbf{y}$  to be linearly independent.

*See Definition 4.9 in the textbook.*



(c) Using the definition from part (b), prove that if  $a \neq b$ , then  $\mathbf{x}$ ,  $\mathbf{y}$  are linearly independent.

*Suppose*

$$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}.$$

*We need to show that  $c_1$  and  $c_2$  are 0. Let  $\mathbf{z} = c_1\mathbf{x} = -c_2\mathbf{y}$ . Then  $L(\mathbf{z}) = L(c_1\mathbf{x}) = c_1L(\mathbf{x}) = c_1a\mathbf{x} = a\mathbf{z}$ . Similarly,  $L(\mathbf{z}) = b\mathbf{z}$ . Thus,  $(a - b)\mathbf{z} = \mathbf{0}$ . Since  $a \neq b$ , we must have  $\mathbf{z} = \mathbf{0}$ . Thus,  $c_1\mathbf{x} = \mathbf{0}$ . But  $\mathbf{x} \neq \mathbf{0}$ , so we must have  $c_1 = 0$ . The same argument gives  $c_2 = 0$  and so  $\mathbf{x}, \mathbf{y}$  are linearly independent.*

[END.]