# MATH 2310 QUIZ

Friday 2 October 2009. You have 50 minutes. No calculators are permitted. Please show all working.

(1) True or False? (Explain your answer.)

(a) No linear system has exactly two solutions.

True: a linear system has either 0, 1 or infinitely many solutions.

(b) If $A$ is *any* $n \times n$ matrix, then the matrix $I_n - AA^T$ is symmetric. (Here, $I_n$ denotes the identity matrix of size $n \times n$.)

True: if $A$ is any $n \times n$ matrix, then $(I_n - AA^T)^T = (I_n)^T - (AA^T)^T = I_n - (A^T)^T A^T = I_n - AA^T$ according to the rules about transposes. So $I_n - AA^T$ is symmetric.

(2) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate each of the following. If it is not defined, say so.

(a) $A(C + B^T)$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left( \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 6 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \end{bmatrix}$$

(b) $B^{-1}$

Not defined because $B$ is not square.

(3) Let $X = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

(a) Show that $X^2 - 7X + I_2 = 0$.

$$X^2 - 7X + I_2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - 7\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} - \begin{bmatrix} 14 & 21 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Determine whether $X$ is invertible, and find $X^{-1}$ if it exists.

$$\begin{bmatrix} X & I_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow_{r_2-(3/2)r_1} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{bmatrix}$$

$$\longrightarrow_{r_1-6r_2} \begin{bmatrix} 2 & 0 & 10 & -6 \\ 0 & 1/2 & -3/2 & 1 \end{bmatrix}$$

$$\longrightarrow_{(1/2)r_1} \begin{bmatrix} 1 & 0 & 5 & -3 \\ 0 & 1/2 & -3/2 & 1 \end{bmatrix}$$

$$\longrightarrow_{2r_2} \begin{bmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

So $X$ is invertible and $X^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$.

(4) Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 9 \\ 3 & 1 & 12 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Find all solutions to the linear system $A\mathbf{x} = \mathbf{b}$.

The augmented matrix of this system is

$$A = \begin{bmatrix} 2 & 1 & 3 & \vdots & 1 \\ 1 & 0 & 9 & \vdots & 1 \\ 3 & 1 & 12 & \vdots & 1 \end{bmatrix}$$

Perform row operations:

$$\to_{r_3-r_1} \begin{bmatrix} 2 & 1 & 3 & \vdots & 1 \\ 1 & 0 & 9 & \vdots & 1 \\ 1 & 0 & 9 & \vdots & 0 \end{bmatrix}$$

$$\to_{r_3-r_2} \begin{bmatrix} 2 & 1 & 3 & \vdots & 1 \\ 1 & 0 & 9 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & -1 \end{bmatrix}$$

We have obtained the equation $0 = -1$, which has no solutions. Thus, the original system also has not solutions.

(b) Is $A$ invertible? Explain why or why not.

$A$ is not invertible because there is some $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution, by part (a).

(5) Determine whether the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

is invertible, and find $A^{-1}$ if it exists.

Form the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

and reduce it to rref as follows:

$$\to_{2r_2} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -2 & 4 & -4 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\to_{r_2+r_1} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -4 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow_{r_2+3r_3} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 3 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow_{\text{swap } r_2, r_3} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow_{r_2-r_3} \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 2 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow_{r_1-r_2} \begin{bmatrix} 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & -1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 2 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow_{(1/2)r_1, -r_2, (1/2)r_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1/2 & 1 & 3/2 \end{bmatrix}$$

Conclusion: $A$ is invertible (because the rref of $A$ is $I$) and

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1/2 & 1 & 3/2 \end{bmatrix}$$

[**END**]