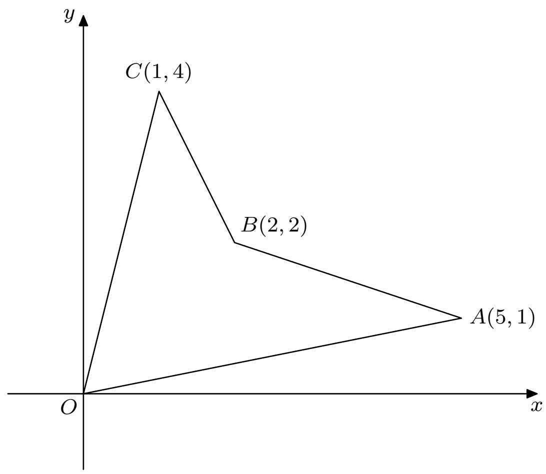
Solutions to Math 2310 Take-home prelim 2

Question 1. Find the area of the quadrilateral $OABC$ on the figure below, coordinates given in brackets. [See pp. 160—163 of the book.]



Solution.

$$\text{Area}(OABC) = \text{Area}(OAB) + \text{Area}(OBC) = \tfrac{1}{2}|\det(\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix})| + \tfrac{1}{2}|\det(\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix})| = 4 + 3 = 7.$$

Question 2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 1 \end{bmatrix}$$

(a) Calculate the nullspace of the matrix A.

(b) Let $B = A^T$. Find the rank of B.

(c) Find a basis for the column space of B.

Solution.

(a) The nullspace of A is the set of solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the linear system $A\mathbf{x} = \mathbf{0}$. Use Gauss-Jordan reduction to solve it. The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Setting $x_2 = r$, $x_4 = s$, we have $x_1 = -2r + 3s$, $x_3 = -s$, or equivalently

$$\mathbf{x} = \begin{bmatrix} -2r + 3s \\ r \\ -s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} s, \quad \text{where } r \text{ and } s \text{ are free.}$$

(b) $\operatorname{rank} B = \operatorname{rank} A^T = \operatorname{rank} A = 2$ since the rows of A are linearly independent.

(c) Since the columns of matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \\ 0 & 1 \end{bmatrix}$$

are linearly independent, they form a basis of it's column space.

Question 3. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

(a) Find the reduced row echelon form of A.

(b) Do the rows of A span $\mathbb{R}_3$? Explain your answer.

(c) Do the columns of A span $\mathbb{R}^3$? Explain your answer.

(d) Your friend Bob claims that there exist bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbb{R}^3$ such that $[\mathbf{x}]_S = A[\mathbf{x}]_T$ for all $\mathbf{x}$ in $\mathbb{R}^3$. Explain why this cannot possibly be true.

Solution.

(a) The reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

.

(b) The row space of A equals the row space of R, which is 2-dimensional. Hence rows of A do not span $\mathbb{R}_3$.

(c) No, because the dimension of the column space of A is equal to that of the row space, which is 2.

(d) Suppose, this is true. Since $\operatorname{rank} A = 2 < 3$, A is singular. Therefore there is a nonzero 3-vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ such that $A\mathbf{u} = \mathbf{0}$. Let $\mathbf{x} = u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + u_3\mathbf{w}_3$. Then $\mathbf{x} \neq \mathbf{0}$ (because T is linearly independent and $\mathbf{u} \neq 0$) and $[\mathbf{x}]_T = \mathbf{u}$. We have $[\mathbf{x}]_S = A[\mathbf{x}]_T = A\mathbf{u} = \mathbf{0}$, which means $\mathbf{x} = \mathbf{0}$. Contradiction.

Question 4. Let A be an $n \times n$ matrix with integer entries.

(a) If $\det(A) = 1$, show that A^{-1} has integer entries.

(b) Suppose A^{-1} has integer entries. What are the possibilities for $\det(A)$? Explain.

Solution.

(a) By the formula of the inverse matrix, (i, j) entry of A^{-1} is $\frac{A_{ji}}{\det(A)}$. If $\det(A) = 1$, this entry equals A_{ji}, which is an integer. Indeed, up to sign, the cofactor A_{ji} is the determinant of a submatrix of A, and the determinant of a matrix with integer entries is an integer.

(b) $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$. So, the product of integers $\det(A)$ and $\det(A^{-1})$ equals 1, which implies $\det(A) = \pm 1$. These possibilities are actually realized, e. g. $A = I_n$ ($\det(A) = 1$) or the diagonal matrix with entries $-1, 1, 1, \ldots, 1$ on the diagonal ($\det(A) = -1$).

Question 5. Find out whether the matrices

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

form a basis in the space of all 2×2 matrices.

Solution.

Check linear independence. The equation $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 = 0$ is equivalent to a homogeneous linear system $A\mathbf{x} = 0$, where

$$A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Zero out below $(1, 1)$ entry using elementary row transformations of the third type:

$$B = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & -7 & -2 & -1 \\ 0 & -10 & -8 & -2 \\ 0 & -13 & -10 & -7 \end{bmatrix}.$$

We have $\det(A) = \det(B)$. By the cofactor expansion in the first column of B, we have $\det(B) = B_{11} = -160 \neq 0$. Then A is invertible and the only solution is $x_1 = x_2 = x_3 = x_4 = 0$. Thus, matrices $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent. Since their number (four) equals the dimension of the space of 2×2 matrices, they form a basis in that space.

Question 6. Find all vectors in $\mathbb{R}^3$ of length ≤ 2 with integer entries. Which of them are orthogonal to the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$?

Solution.

Denote the set of integer vectors of length ≤ 2 by S. The vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to S if and only if

$$x^2 + y^2 + z^2 \leq 2^2 = 4.$$

If x, y, z are integers, their absolute values can be equal to 0, 1 or 2 (otherwise the length will be bigger than 2). All the vectors in S satisfying $0 \le x \le y \le z$ are

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}.$$

The rest of S is obtained from these by permutating entries and multiplying some of them by -1. [Because any integer vector in S can be transformed to one with $0 \le x \le y \le z$ by such operations.] In total, S has 33 elements.

Orthogonality to $\mathbf{v}$ means

$$x + y + 2z = 0.$$

All vectors in S satisfying to this relation are

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}.$$

Question 7. The population of sapsuckers in Sapsucker Woods is described by the following model. Let c_k denote the number of chicks in year k, let j_k denote the number of juveniles in year k, and let a_k denote the number of adults in year k. Then

$$\begin{bmatrix}c_{k+1}\\j_{k+1}\\a_{k+1}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0.2\\0.25 & 0.875 & 0\\0 & 0.5 & 0.8\end{bmatrix}\begin{bmatrix}c_k\\j_k\\a_k\end{bmatrix}$$

Let A be the matrix

$$A = \begin{bmatrix}0 & 0 & 0.2\\0.25 & 0.875 & 0\\0 & 0.5 & 0.8\end{bmatrix}$$

(a) A vector $\mathbf{v}$ in $\mathbb{R}^3$ is called a *steady-state vector* of A if $A\mathbf{v} = \mathbf{v}$. Explain what this means in terms of the model.

(b) Find all steady-state vectors for A.

(c) After heavy logging in Sapsucker woods, biologists find that the model is no longer accurate. Instead, a more suitable model is

$$\begin{bmatrix}c_{k+1}\\j_{k+1}\\a_{k+1}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0.2\\0.25 & 0 & 0\\0 & 0.5 & 0\end{bmatrix}\begin{bmatrix}c_k\\j_k\\a_k\end{bmatrix}$$

Under this new model, what do you think will happen to the population of sapsuckers in the long term? Explain your answer.

Solution.

(a) This means that the population is in balance: $c_k = c$, $j_k = j$, $a_k = a$ are constants (do not depend on k).

(b) The equation $A\mathbf{v} = \mathbf{v}$ is equivalent to the linear system $(I_3 - A)\mathbf{v} = \mathbf{0}$. The reduced row echelon form of

$$I_3 - A = \begin{bmatrix} 1 & 0 & -0.2 \\ -0.25 & 0.125 & 0 \\ 0 & -0.5 & 0.2 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 & -0.2 \\ 0 & 1 & -0.4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $a = 5r$, we have $\mathbf{v} = \begin{bmatrix} c \\ j \\ a \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, where r is free.

(c) Under the new model, the population will be extinct. Indeed, set $t_k = c_k + j_k + a_k$ (the total of all species in year k). Then $0 \le t_{k+1} = c_{k+1} + j_{k+1} + a_{k+1} = 0.2a_k + 0.25c_k + 0.5j_k \le 0.5(a_k + c_k + j_k) = 0.5t_k$. Thus $0 \le t_{k+1} \le 0.5t_k \le 0.5^2 t_{k-1} \le \cdots \le 0.5^k t_1$. Since the limit of $0.5^k t_1$ as $k \to \infty$, is 0, so is the limit of t_k.

Question 8. Let

$$A = \begin{bmatrix} 3 & 5 & 7 & 3 & 2 \\ 2 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 5 & 2 \end{bmatrix}$$

1. Calculate $\det(A)$.
2. Is A invertible? Explain your answer.
3. Calculate $\det(AA^T)$.

Solution. **(a)**

$$\begin{vmatrix} 3 & 5 & 7 & 3 & 2 \\ 2 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 7 & 3 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 5 & 2 \end{vmatrix} = 0.$$

Here, the first identity is obtained by subtracting row 3 from row 2, the second identity follows from the fact that the determinant of a matrix having two equal rows is 0.

(b) No, A is singular since $\det(A) = 0$.

(c) $\det(AA^T) = \det(A)\det(A^T) = 0$.