4130 HOMEWORK QS

Due never

Solutions to these problems will not be supplied (but see one of us if you want to know how to do a particular problem). These problems are not necessarily representative of what will be on the exam, but they are left over from a list of problems which I did not have time to put into any of the homeworks or exams. Another good source of problems is anything which you were asked to prove as an "exercise" in the lecture notes! Also, there are plenty of problems in the textbook.

1. Standard problems

(1) Let $\{x_n\}$ be a convergent sequence and $k \geq 0$. Show that $\{x_{n+k}\}$ also converges.

(2) Show that an unbounded monotone sequence either diverges to ∞ or diverges to $-\infty$.

(3) Show that if a sequence $\{x_n\}$ has a finite liminf and limsup, then it is a bounded sequence.

(4) Give an example of open sets U, V such that $U \neq V$ but $U \cap \mathbb{Q} = V \cap \mathbb{Q}$.

(5) Using the intermediate value theorem, show that the continuous image of an interval is an interval.

(6) Show that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$.

(7) Suppose $f : (a, b) \to \mathbb{R}$ is a continuous bounded strictly increasing function. Show that the image of f is the interval (c, d) where $c = \inf_{x\in(a,b)} f(x)$ and $d = \sup_{x\in(a,b)} f(x)$.

(8) Let $f : A \to \mathbb{R}$ be continuous, where $A \subset \mathbb{R}$. Suppose $x_0 \in A$ and $f(x_0) < 0$. Show that there is a neighbourhood U of x_0 and a natural number $N \in \mathbb{N}$ such that $f(y) \leq -1/N$ for all $y \in U$.

(9) Consider the power series

$$f(x) = \sum_{p\in P} x^p$$

where $P = \{2, 3, 5, 7, \ldots\}$ denotes the set of prime numbers. What is the radius of convergence of $f(x)$? Does the series $f(1)$ converge? Does the series $f(-1)$ converge? Show that for $x \geq 0$, $f(x) \leq x^2/(1-x)$. What is $f^{(10000)}(0)$?

(10) Let b_n be any sequence with a finite limsup and let c_n be a sequence which converges to $c > 0$. Prove that

$$\limsup_n (b_n c_n) = c \limsup_n b_n.$$

(11) Find the radius of convergence R of the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

Prove that

$$\exp(f(x)) = \frac{1}{1-x}$$

if $|x| < R$.

(12) Consider the series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n + 2^n} x^n.$$

(a) Show that the radius of convergence of $f(x)$ is $R = 2$.

(b) Show that $f(x)$ defines a C^∞ function on $(0, 2)$ which has an inverse function $g : f((0, 2)) \to (0, 2)$.

(c) Show that g is C^1 and that

$$g'(x) = \frac{1}{f'(g(x))}.$$

2. Extra problems, mostly harder

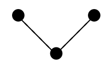
(13) Let $\{x_n\}$ be a Cauchy sequence of rational numbers. For $S \subset \mathbb{N}$, define

$$x_n^S = \begin{cases} x_n + 1 & n \in S \\ x_n & n \notin S. \end{cases}$$

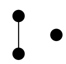
Find all $S \subset \mathbb{N}$ such that x_n^S is not a Cauchy sequence.

(14) A partially ordered set $(X, \leq)$ is said to be *complete* if every subset $S \subset X$ which is bounded above has a least upper bound. Which of the posets defined by the following Hasse diagrams are complete? (Note: Hasse diagrams are not an examinable topic. You can ignore this question if you have never seen them before.)

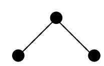
(a)



(b)



(c)



(15) Let F be an ordered field and $x \in F \setminus \{0\}$. Show that $x > 0$ if and only if $x^{-1} > 0$.

(16) Define a sequence by

$$x_n = x_{n-1} + x_{n-2}$$

for $n \geq 2$ and some $x_1, x_2 \in \mathbb{R}$. Show that if the sequence converges to a limit L, then $L = 0$. Is it possible to have a nonzero x_1, x_2 such that the sequence converges? (Warning: the second question is quite hard.)

(17) If $|A| = |B|$, show that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

(18) One-sided derivatives. (5.4.6 Ex 4).

(19) Suppose I am standing at a railroad station and a train passes me traveling at 20 mph. Five minutes later, the train passes me again, still traveling at 20 mph. I reason that if $f(t)$ denotes the distance between me and the train, then $f(0) = f(5) = 0$, so by Rolle's Theorem there must exist a time t_0 with $0 < t_0 < 5$ and $f'(t_0) = 0$, ie. the train had speed zero, and so was stationary. So the train must have stopped at some point. What is the flaw in this reasoning?

(20) If $f : A \to \mathbb{R}$ is continuous and monotone increasing and $[a, b] \subset A$, show that $f([a, b]) = [f(a), f(b)]$.

(21) Two scientists, A and B, propose competing theories for the energy distribution $f(\lambda)$ corresponding to the wavelength λ in some physical problem. Scientist A's formula

is

$$f_A(\lambda) = \frac{1}{\lambda^4}$$

while Scientist B's formula is

$$f_B(\lambda) = \lambda^{-5} e^{-1/\lambda}.$$

Scientist B writes a paper saying that Scientist A's formula is false because, although it agrees well with experimental results for large λ, its predictions are completely wrong for small λ (in fact, $f(\lambda)$ is known experimentally to tend to zero as $\lambda \to 0$, while $f_A(\lambda)$ tends to infinity). Scientist A responds with a furious diatribe against Scientist B's formula, claiming that although $f_B(\lambda)$ gives good predictions for small λ, it is completely inaccurate for large λ.

Scientist C then settles the dispute by discovering that

$$f(\lambda) = \frac{\lambda^{-5}}{e^{1/\lambda} - 1}.$$

Explain why A and B were both right.

3. Practice final

Math 413 final exam, 13 May 2008. The exam starts at 9:00 am and you have 150 minutes. No textbooks or calculators may be used during the exam. **This exam is printed on both sides of the paper.** *Good luck!*

(1) **(20 marks)** Let $X = (0, 1] \subset \mathbb{R}$. State whether each of the following statements about X is true or false, giving a brief reason for each answer.

(a) X is bounded.

(b) X can be written as a countable union of open sets.

(c) X is compact.

(d) There is a point $x_0 \in X$ at which the function $f(x) = \log(x) + x^5 - 8x^4 - 3$ achieves its supremum on X (that is, $f(x_0) = \sup\{f(x) : x \in X\}$).

(2) **(20 marks)** Let $A \subset \mathbb{R}$. Recall that a function $f : A \to \mathbb{R}$ is said to satisfy a Lipschitz condition on A if there is some $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Let $n \in \mathbb{N}$. Show that the function $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = \sqrt{x + \frac{1}{n}}$ satisfies a Lipschitz condition on $[0, 1]$.

(Hint: you may wish to use the fact that for all $a, b > 0$, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.)

(b) Show that the sequence of functions $\{f_n\}$ converges uniformly on $[0, 1]$ to the function $f(x) = \sqrt{x}$.

(c) Show that $f(x) = \sqrt{x}$ does *not* satisfy a Lipschitz condition on $[0, 1]$.

(d) Now suppose $A \subset \mathbb{R}$ and $f_n : A \to \mathbb{R}$ are functions such that there exists $M \in \mathbb{R}$ such that $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $n \in \mathbb{N}$ and all $x, y \in A$. Suppose the sequence of functions $\{f_n\}$ converges uniformly on A to a function $f : A \to \mathbb{R}$. Show that f satisfies a Lipschitz condition on A. Why does this not contradict your answer to part (c)?

[TURN OVER]

(3) **(20 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) State what it means for f to be uniformly continuous on $\mathbb{R}$.

(b) State the Mean Value Theorem.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that the derivative f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$.

(d) Show that $f(x) = e^{-x^2}$ is uniformly continuous on $\mathbb{R}$.

(4) **(20 marks)** Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(a) State what it means for f to be Riemann integrable.

(b) Show that if $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then so is $f + g$.

(c) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is *not* Riemann integrable.

(d) Now let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. Define $F(x) = \int_0^1 f(x + t)dt$. Show that F is continuous on $\mathbb{R}$.

(5) **(20 marks)** Consider the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1}.$$

(a) Prove that the series converges absolutely and uniformly on $[-a, a]$ for all $a > 0$. Deduce that this power series defines a C^∞ function $f : \mathbb{R} \to \mathbb{R}$.

(b) Prove that

$$f(x) + f'(x) + f''(x) + f'''(x) = e^x$$

for all $x \in \mathbb{R}$.

(c) Show that $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

(d) Show that $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

[END.]