MATH 4130 FINAL EXAM

Math 4130 final exam, 18 May 2010. The exam starts at 7:00 pm and you have 150 minutes. No textbooks or calculators may be used during the exam. **This exam is printed on both sides of the paper.** *Good luck!*

(1) **(20 marks.)** Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Let $L \in \mathbb{R}$.

(a) Explain what it means to say $\lim_{n\to\infty} x_n = L$.

(b) Explain what is meant by $\limsup_n y_n$.

(c) Show that if $\lim_{n\to\infty} x_n = L$ then $\lim_{n\to\infty} |x_n| = |L|$.

(d) Suppose $\limsup_n y_n = L$. Is it necessarily true that $\limsup_n |y_n| = |L|$? Explain your answer.

(2) **(20 marks.)** A real number α is said to be *algebraic* if for some $n \in \mathbb{N}$ there is a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree n with $a_i \in \mathbb{Q}$ for all i, and with $f(\alpha) = 0$. (In this case, we say that α is a *root* of f.) If α is not algebraic, it is said to be *transcendental.*

(a) Show that the set of all algebraic numbers is countable. (You may use without proof the fact that a polynomial of degree n has at most n roots.)

(b) Show that there exists a transcendental number.

(c) Now consider the expression $g(x) = \sum_{n=1}^{\infty} x^{n!}$. Show that the series defines a C^∞ function $g : (-1, 1) \to \mathbb{R}$. [Remark: the number $g(1/10)$ is known to be transcendental. Do not prove this!]

(3) **(20 marks.)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) State what it means for f to be uniformly continuous on $\mathbb{R}$.

(b) State the Mean Value Theorem.

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that the derivative f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$.

(d) Show that $f(x) = \log(1 + x^2)$ is uniformly continuous on $\mathbb{R}$. [TURN OVER.]

(4) **(20 marks.)** Recall that for $x > 0$ and $a \in \mathbb{R}$, we define $x^a = \exp(a \log(x))$.

(a) Let $a \in \mathbb{R}$. Show that $\frac{d}{dx}(x^a) = ax^{a-1}$.

(b) Let $a > 1$. Show that

$$\int_1^N \frac{1}{x^a} dx = \frac{1}{1-a}(N^{1-a} - 1).$$

(c) Let I be a closed interval. Explain what is meant by the *upper and lower Riemann sums* $S^+(f, P)$ and $S^-(f, P)$ of a continuous function $f : I \to \mathbb{R}$ with respect to a partition P of I.

(d) For $N \geq 2$ and $a > 1$, show that

$$\sum_{n=2}^{N} \frac{1}{n^a} \leq \int_1^N \frac{1}{x^a} dx.$$

(e) Show that if $a > 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges.

(5) **(20 marks.)** The following problem is set in an analysis exam which you are grading:

Problem: (10 marks) Suppose $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. Let x be a cluster point of A. Suppose $\lim_{x \to a} f(x) = L \neq 0$. Show that $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}$.

A student writes the following solution:

"**My solution:**

$$\left| \frac{1}{f(x)} - \frac{1}{L} \right| = \left| \frac{L - f(x)}{f(x)L} \right| = \frac{|f(x) - L|}{|f(x)||L|} < \frac{\varepsilon}{|f(x)||L|}$$

if $|f(x) - L| < \varepsilon$.

So given $\varepsilon > 0$, choose $\delta > 0$ such that, if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon \cdot \inf |f(x)| \cdot |L|$. Then

$$|x - a| < \delta \implies \left| \frac{1}{f(x)} - \frac{1}{L} \right| < \varepsilon.$$

QED."

(a) Comment on any aspects of the solution which you think are incorrect, or which could be improved.

(b) How many marks (out of a maximum possible 10) would you award the student? Explain your answer.

[END.]