MATH 4130 FINAL EXAM

Math 4130 final exam, 18 May 2010. The exam starts at 7:00 pm and you have 150 minutes. No textbooks or calculators may be used during the exam. **This exam is printed on both sides of the paper.** *Good luck!*

(1) **(20 marks.)** Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Let $L \in \mathbb{R}$.

(a) Explain what it means to say $\lim_{n\to\infty} x_n = L$.

It means that for all $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that if* $n > N$ *then* $|x_n - L| < \varepsilon$.

(b) Explain what is meant by $\limsup_n y_n$.

One definition: $\limsup_n y_n$ *is the supremum of the set of limit-points (limits of subsequences) of the sequence* $\{y_n\}$. *Another definition:* $\limsup_n y_n$ *is the limit of the sequence* $\{\sup_{k\geq n} y_k\}$ *as* $n \to \infty$.

(c) Show that if $\lim_{n\to\infty} x_n = L$ then $\lim_{n\to\infty} |x_n| = |L|$.

Suppose $\lim_{n\to\infty} x_n = L$. *Let* $\varepsilon > 0$. *Then there exists* $N \in \mathbb{N}$ *such that if* $n > N$ *then* $|x_n - L| < \varepsilon$. *If* $n > N$ *then* $||x_n| - |L|| \leq |x_n - L| < \varepsilon$.

(d) Suppose $\limsup_n y_n = L$. Is it necessarily true that $\limsup_n |y_n| = |L|$? Explain your answer.

No. For example, take the sequence $0, -1, 0, -1, \ldots$ *for* $\{y_n\}$. *Then* $\limsup_n y_n = 0$ *but* $\limsup_n |y_n| = 1$.

(2) **(20 marks)** A real number α is said to be *algebraic* if for some $n \in \mathbb{N}$ there is a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree n with $a_i \in \mathbb{Q}$ for all i, and with $f(\alpha) = 0$. (In this case, we say that α is a *root* of f.) If α is not algebraic, it is said to be *transcendental*.

(a) Show that the set of all algebraic numbers is countable. (You may use without proof the fact that a polynomial of degree n has at most n roots.)

The set of all algebraic numbers is the union

$$\bigcup_{n\geq 1}\bigcup_{p\in P_n}(\textit{roots of } p).$$

Where P_n denotes the set of all polynomials with rational coefficients of degree n. The set P_n is in bijection with $\mathbb{Q}^n$, a countable set. We see that the set of algebraic numbers is a countable union of countable sets, so it is countable.

(b) Show that there exists a transcendental number.

Since $\mathbb{R}$ is uncountable, it cannot be equal to the set of algebraic numbers. So there must be a real number which is not algebraic.

(c) Now consider the expression $g(x) = \sum_{n=1}^{\infty} x^{n!}$. Show that the series defines a C^∞ function $g : (-1, 1) \to \mathbb{R}$. [Remark: the number $g(1/10)$ is known to be transcendental. Do not prove this!]

This is a power series $g(x) = \sum a_n x^n$ with coefficients

$$a_n = \begin{cases} 1 & n = k! \\ 0 & \textit{otherwise.} \end{cases}$$

We see that $\limsup_n |a_n|^{1/n} = 1$ *and so the power series has radius of convergence 1. Therefore, by theorems on power series, it defines a C^∞ function on the interval $(-1, 1)$.*

(3) **(20 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) State what it means for f to be uniformly continuous on $\mathbb{R}$.

It means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

(b) State the Mean Value Theorem.

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and that the derivative f' is bounded. Show that f is uniformly continuous on $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose there exists $M > 0$ with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Then if $a < b$, then $\frac{f(b)-f(a)}{b-a} = f'(x_0) \leq M$ for some $x_0 \in (a, b)$. So $|f(b) - f(a)| \leq M|b - a|$. Therefore, given $\varepsilon > 0$, if $\delta < \varepsilon/M$ then $|b - a| < \delta$ implies $|f(b) - f(a)| < \varepsilon$. So f is uniformly continuous.

(d) Show that $f(x) = \log(1 + x^2)$ is uniformly continuous on $\mathbb{R}$. [TURN OVER.]

In view of the previous problem, it suffices to show that the derivative of f is bounded. So it suffices to show that $\frac{2|x|}{1+x^2}$ is bounded. If $|x| \geq 1$ then $\frac{2|x|}{1+x^2} \leq \frac{2}{|x|} \leq 2$ while if $|x| \leq 1$ then also $\frac{2|x|}{1+x^2} \leq 2|x| \leq 2$.

(4) **(20 marks.)** Recall that for $x > 0$ and $a \in \mathbb{R}$, we define $x^a = \exp(a \log(x))$.

(a) Let $a \in \mathbb{R}$. Show that $\frac{d}{dx}(x^a) = ax^{a-1}$.

We use the chain rule to differentiate $e^{a\log(x)}$. This gives $\frac{a}{x}e^{a\log(x)} = ae^{-\log(x)}e^{a\log(x)} = ae^{(a-1)\log(x)}$.

(b) Let $a > 1$. Show that

$$\int_1^N \frac{1}{x^a}dx = \frac{1}{1-a}(N^{1-a} - 1).$$

By the previous problem, the derivative of $\frac{1}{1-a}x^{1-a}$ is x^{-a}. Therefore, by the fundamental theorem of calculus, we have

$$\int_1^N \frac{1}{x^a}dx = \frac{1}{1-a}x^{1-a}|_1^N.$$

(c) Let I be a closed interval. Explain what is meant by the *upper and lower Riemann sums* $S^+(f, P)$ and $S^-(f, P)$ of a continuous function $f : I \to \mathbb{R}$ with respect to a partition P of I.

Let $P = \{x_0 < x_1 < \cdots < x_n\}$ be a partition. The upper Riemann sum $S^+(f, P)$ is the sum

$$\sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}).$$

The lower Riemann sum $S^-(f, P)$ is the sum

$$\sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}).$$

(d) For $N \geq 2$ and $a > 1$, show that

$$\sum_{n=2}^{N} \frac{1}{n^a} \leq \int_1^N \frac{1}{x^a} dx.$$

The function $f(x) = x^{-a}$ is decreasing, because its derivative is $-ax^{a-1}$, which is $-a$ times the exponential of something, which must be negative. So the sum on the left hand side is the lower Riemann sum for the function f on $[1, N]$ with respect to the partition $P = \{1, 2, \ldots, N\}$. The lower Riemann sum is $\leq$ the integral since the integral is the supremum of the set of lower Riemann sums of all partitions P.

(e) Show that if $a > 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges.

The N^{th} partial sum of the series is bounded above by the integral, whose value is

$$\int_1^N \frac{1}{x^a} dx = \frac{1}{1-a}(N^{1-a} - 1).$$

Since $a > 1$, the sequence

$$b_N = \frac{1}{1-a}(N^{1-a} - 1)$$

converges, and so is bounded. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^a}$ is an increasing bounded sequence, so it converges.

(5) **(20 marks.)** The following problem is set in an analysis exam which you are grading:

Problem: (10 marks) Suppose $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. Let x be a cluster point of A. Suppose $\lim_{x \to a} f(x) = L \neq 0$. Show that $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}$.

A student writes the following solution:

"**My solution:**

$$\left| \frac{1}{f(x)} - \frac{1}{L} \right| = \left| \frac{L - f(x)}{f(x)L} \right| = \frac{|f(x) - L|}{|f(x)||L|} < \frac{\varepsilon}{|f(x)||L|}$$

if $|f(x) - L| < \varepsilon$.

So given $\varepsilon > 0$, choose $\delta > 0$ such that, if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon \cdot \inf |f(x)| \cdot |L|$. Then

$$|x - a| < \delta \implies \left| \frac{1}{f(x)} - \frac{1}{L} \right| < \varepsilon.$$

QED."

(a) Comment on any aspects of the solution which you think are incorrect, or which could be improved.

The proof is OK except for $|f(x)-L| < \varepsilon \cdot \inf|f(x)| \cdot |L|$. *There is not necessarily any such thing as* $\inf|f(x)|$. *No set is specified over which the infimum is taken. The student should have shown that* f *is bounded below near* a. *That is, there exists* δ_1 *such that* $|x-a| < \delta_1$ *implies* $||f(x)|-|L|| \leq |f(x)-L| < |L|/2$ *and then* $|f(x)| \geq |L|-|L|/2 = |L|/2$. *Now replace the* inf *by* $|L|/2$ *in the above proof, and replace* δ *by the minimum of* δ *and* δ_1. *Then the proof works.*

(b) How many marks (out of a maximum possible 10) would you award the student? Explain your answer.

The proof is mostly, but not wholly, correct. Therefore, any answer < 10 *is acceptable here.*

[END.]