MATH 4130 HONORS INTRODUCTION TO ANALYSIS I. PRELIM 1. THURSDAY MARCH 11 2010

Please attempt all questions. You have 70 minutes. You may use any theorems from the lecture notes, but please clearly state any theorems which you use.

(1) **(9 marks)** Let $X = (0,1) \cup [3,4] \subset \mathbb{R}$. State whether the following statements about X are true or false and give a brief reason in each case.

(a) $\sup(X) = 4$.

True, since 4 *is clearly an upper bound and any upper bound is* ≥ 4, *by definition.*

(b) X can be written as a union of open sets.

False. Any union of open sets is open, and X *is not open.*

(c) $|X| = |\mathbb{R}|$.

True. $|X| \leq |\mathbb{R}|$ *because* $X \subset \mathbb{R}$. *Also, in the lectures we have constructed an injection* $\mathcal{P}(\mathbb{N}) \to (0,1)$. *Therefore,* $|X| \geq |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

(2) **(19 marks)** Let $\{x_n\}$ be a sequence of real numbers.

(a) **(3 marks)** State what it means for $\{x_n\}$ to *converge* to the limit $L \in \mathbb{R}$.

It means that for all $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that if* $n > N$ *then* $|x_n - L| < \varepsilon$.

(b) **(8 marks)** Let $k \in \mathbb{N}$ and define a sequence $\{y_n\}$ by $y_n = x_{n+k}$, $n \geq 1$. Suppose $\{x_n\}$ converges to L. Show that $\{y_n\}$ also converges to L.

Let $\varepsilon > 0$. *Then there exists* $N \in \mathbb{N}$ *such that if* $n > N$ *then* $|x_n - L| < \varepsilon$. *Then if* $n > N$, *we have* $n + k > N$ *and so* $|y_n - L| = |x_{n+k} - L| < \varepsilon$, *which proves that* $\{y_n\} \to L$.

(c) **(8 marks)** Let $x_1 \in \mathbb{R}$ and define a sequence of real numbers $\{x_n\}$ by

$$x_{n+1} = x_n^2 + x_n + 1, \qquad n \geq 1.$$

Show that the sequence $\{x_n\}$ does not converge.

Here are two ways:

Firstly, we know from the lectures that if $\{x_n\} \to L$ *then* $\{x_n^2 + x_n + 1\} \to L^2 + L + 1$ *(see homework 4, or Theorem 2.4.2 in the textbook). Also, from part (b),* $\{x_{n+1}\} \to L$*. Therefore, we have* $L = L^2 + L + 1$ *and* $L^2 + 1 = 0$*. But no real number* L *has* $L^2 + 1 = 0$*. Therefore, the sequence* $\{x_n\}$ *cannot converge.*

Alternative method: we have $x_{n+1} > x_n + 1$*. Thus* $x_{n+1} - x_n > 1$ *for all* $n \geq 1$*. Therefore, the sequence* $\{x_n\}$ *cannot be Cauchy and so does not converge.*

(3) **(22 marks)** Let $A \subset \mathbb{R}$.

(a) **(3 marks)** Explain what it means to say that $x \in \mathbb{R}$ is a *cluster point* (a.k.a. limit-point; accumulation point) of A.

x *is a cluster point of* A *if every open set* U *with* $x \in U$ *also contains infinitely many points of* A*.*

(b) **(3 marks)** Explain what it means to say that the set A is *bounded.*

A *is bounded if there exists* $N \in \mathbb{N}$ *with* $A \subset [-N, N]$*.*

(c) Now let

$$S = \{x \in \mathbb{R} : x \text{ is a cluster point of } A\}.$$

(i) **(8 marks)** Show that S is a closed set.

We will show that S *contains all of its cluster points. By a theorem from the lectures, this is equivalent to being closed.*

Let $x \in \mathbb{R}$ *be a cluster point of* S*. Let* U *be an open set with* $x \in U$*. Then* U *contains infinitely many points of* S*. In particular,* U *contains a cluster point* y *of* A*. But then, by definition of a cluster point,* U *contains infinitely many points of* A*. So every open set containing* x *also contains infinitely many points of* A*, and therefore* x *is a cluster point of* A*, so* $x \in S$*, as desired.*

(ii) **(8 marks)** Suppose A is bounded. Show that S is a compact set.

Recall that compactness is the same as being closed and bounded. We showed in part (a) that S *is closed. Thus, we need to show that* S *is bounded.*

Since A is bounded, we have $A \subset [-N, N]$ for some $N \in \mathbb{N}$. Let $x \in S$. Then $(x-1, x+1)$ contains some $a \in A$. Therefore, $x-1 < a < x+1$ and so $a-1 < x < a+1$, and $|x| \leq |a|+1 \leq N+1$. Thus, $S \subset [-N-1, N+1]$, and so S is bounded.

Alternative method: Observe that $[-N, N]$ is a closed set containing A. Therefore, $[-N, N]$ also contains the closure $\mathrm{cl}(A)$. (By a theorem from the lectures, $\mathrm{cl}(A)$ is the intersection of all the closed sets containing A.) But $\mathrm{cl}(A) = A \cup S$ and so $S \subset [-N, N]$, and therefore S is bounded.

[**END OF PAPER**]