4130 PRELIM 2: TAKE-HOME

Due Tuesday April 20: no extensions. *This is an exam. Unlike a normal homework, you are not allowed to work on these problems in groups. However, do not feel that you have to work in complete isolation: you may discuss the problems with the lecturer or the TA if you need help, or if you find the wording of the questions ambiguous. You are also free to use the textbook, lecture notes, and previous homeworks, but you are not supposed to use any other references.*

(1) Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on A if there exists $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Show that if f satisfies a Lipschitz condition, then f is uniformly continuous.
Suppose f satisfies a Lipschitz condition on A. Then given $\varepsilon > 0$, take $\delta = \varepsilon/M$. If $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. So f is uniformly continuous on A.

(b) Show that the function $f(x) = \sqrt{|x|}$ with domain $[-1, 1]$ is uniformly continuous but does not satisfy a Lipschitz condition.
The function f is uniformly continuous on $[-1, 1]$ because it is continuous, and $[-1, 1]$ is a compact set, so it is uniformly continuous.
To see that f does not satisfy a Lipschitz condition, suppose for a contradiction that there is some M such that $|\sqrt{|x|} - \sqrt{|y|}| \leq M|x - y|$ for all $x, y \in [-1, 1]$. In particular, we can take $y = 0$ and $x > 0$, so $\sqrt{x} \leq Mx$ for all $x > 0$. But then $M \geq \frac{1}{\sqrt{x}}$ for all $x > 0$, which is impossible.

(2) [**Extra credit.**] Recall that for A, B subsets of $\mathbb{R}$, we say that A is *dense* in B if $A \subset B$ and B is a subset of the closure of A.

(a) Give an example of a set A such that $A \cap \mathbb{Q}$ is not dense in A.

Sorry for the confusing notation. Take $A = \{\sqrt{2}\}$ for example. Then $A \cap \mathbb{Q} = \varnothing$ is not dense in A.

(b) Show that $A \subset B$ is dense in B if and only if for every open set U with $U \cap B \neq \varnothing$, we have $U \cap A \neq \varnothing$.

Suppose A is dense in B. Let U be an open set with $U \cap B \neq \varnothing$. We need to show that $U \cap A \neq \varnothing$. Suppose $b \in U \cap B$. Then $b \in \mathrm{cl}(A)$ and so either $b \in A$, or b is a cluster point of A. If $b \in A$, then $b \in U \cap A$ and we are done. If b is a cluster point of A then every open set containing b also contains a point of A. So U contains a point of A and therefore $U \cap A \neq \varnothing$, so we are done.

Conversely, suppose every open set U with $U \cap B \neq \varnothing$ also has $U \cap A \neq \varnothing$. We assume $A \subset B$ and we need to show that B is contained in the closure of A. Let $b \in B$. We must show that either $b \in A$, or b is a cluster point of A. Suppose $b \notin A$. We must show that b is a cluster point of A. Let U be an open set containing b. Then $U \cap B \neq \varnothing$ and so $U \cap A \neq \varnothing$ by hypothesis. So U contains some point $a \in A$, and $a \neq b$ since $b \notin A$. Therefore, every open neighborhood of b contains a point of A different from b, and so b is a cluster point of A, as required.

(c) Let $\xi \in \mathbb{R}$. Show that the set

$$\mathbb{Q}\xi = \{q\xi : q \in \mathbb{Q}\}$$

is dense in $\mathbb{R}$.

This is only true if $\xi \neq 0$.

Assume $\xi > 0$, since $\mathbb{Q}\xi = \mathbb{Q}(-\xi)$. Let (a, b) be an open interval. We must show that (a, b) contains a point of $\mathbb{Q}\xi$. Consider the open interval $(a/\xi, b/\xi)$. This contains a point of $\mathbb{Q}$, so there is some $q \in \mathbb{Q}$ with

$$a/\xi < q < b/\xi$$

whence

$$a < q\xi < b$$

so $q\xi \in \mathbb{Q}\xi \cap (a, b)$. Therefore, every open subset of $\mathbb{R}$ contains a point of $\mathbb{Q}\xi$, and so $\mathbb{Q}\xi$ is dense in $\mathbb{R}$.

(d) Show that every $A \subset \mathbb{R}$ has a countable dense subset.

Let $A \subset \mathbb{R}$. The trick is to consider the set of intervals

$$\mathcal{U} = \{(q, r) : q, r \in \mathbb{Q}, q < r\}.$$

This collection is countable, as we saw in a previous assignment. Write $U_{q,r} = (q, r)$. For each pair of rationals, let $a_{q,r} \in A \cap U_{q,r}$ if $A \cap U_{q,r} \neq \varnothing$. Otherwise, let $a_{q,r}$ be an arbitrary point of A. This gives a collection

$$C := \{a_{q,r} : q, r \in \mathbb{Q}, q < r\}$$

which is countable because it is indexed by a subset of the countable set $\mathbb{Q} \times \mathbb{Q}$. It is also dense in A, since if U is an open set and $U \cap A \neq \varnothing$, then U contains one of the $U_{q,r}$, and so by definition, U contains $a_{q,r}$, and so $U \cap C \neq \varnothing$.

[TURN OVER.]

(3) In this problem, we will show that the Lagrange form of the remainder term in Taylor's Theorem can be derived from the integral form of the remainder.

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Suppose $g(x) \geq 0$ for all $x \in [a, b]$, and that $\int_a^b g(x)dx \neq 0$. Show that there exists $t \in [a, b]$ with

$$\int_a^b f(x)g(x)dx = f(t) \int_a^b g(x)dx.$$

Let M be the supremum of f on $[a, b]$ and let m be the infimum of f on $[a, b]$. These exist because $[a, b]$ is compact, and furthermore there exist $u, y \in [a, b]$ with $f(u) = M$ and $f(y) = m$. Since $g \geq 0$, we have

$$(M - f(x))g(x) \geq 0$$

and

$$(f(x) - m)g(x) \geq 0$$

for all $x \in [a, b]$. From this we conclude that

$$m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$$

Since $\int_a^b g(x)dx \neq 0$, we have

$$f(y) \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq f(u),$$

and so by the Intermediate Value Theorem, there exists $t \in [a,b]$ with

$$f(t) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

which is what we wanted to show.

(b) Let $x_0, x \in \mathbb{R}$. Show that there exists a number t between x and x_0 such that

$$\frac{1}{n!}\int_{x_0}^{x} f(u)(u-x_0)^n du = \frac{f(t)}{(n+1)!}(x-x_0)^{n+1}.$$

If $x > x_0$ then $(u-x_0)^n$ is positive on $[x_0, x]$ and we may apply the previous result. It's an easy calculation to show that

$$f(t)\int_{x_0}^{x}(u-x_0)^n du = \frac{f(t)}{n+1}(x-x_0)^{n+1}.$$

For the case $x < x_0$, we take $g(u) = -(u-x_0)^n$ if n is odd, and $g(u) = (u-x_0)^n$ if n is even, and apply the result from part (a) in the same way.

(4) For $f : [a,b] \to \mathbb{R}$ and $P = \{x_0, \ldots, x_n\}$ a partition of $[a,b]$, define

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

We say that f has *bounded variation* on $[a,b]$ if there exists a number L such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$, $|V(f,P) - L| < \varepsilon$. In this case, we call L the *first-order variation* of f on $[a,b]$ and denote it by $V_a^b(f)$.

(a) If f is monotone increasing on $[a,b]$, show that $V_a^b(f) = f(b) - f(a)$.

If f is monotone increasing on $[a,b]$, then for a partition P, we have

$$V(f,P) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}) = f(b) - f(a).$$

Therefore, we can take the number L to be the common value of all the $V(f,P)$, namely $L = f(b) - f(a)$, and the given definition of having bounded variation is satisfied.

Note: the definition of having bounded variation given in the book is different to the one given in this problem.

(b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which does not have bounded variation.

Take

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q} \end{cases}$$

If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then $V(f, P) = \sum |f(x_i) - f(x_{i-1})|$. In particular, if the x_i are taken to be alternately rational and irrational, then $V(f, P) \geq n - 2$. For any $\delta > 0$, we can therefore take a partition P with $|P| < \delta$ and $V(f, P) \geq N$ for any given natural number N. From this, it is clear that no value of L can exist.

(c) Suppose f is a C^1 function on some open set $U \supset [a, b]$. Show that f has bounded variation on $[a, b]$ and

$$V_a^b(f) = \int_a^b |f'(x)| dx.$$

Since f' is continuous on U, $|f'|$ is continuous on $[a, b]$, so $\int_a^b |f'(x)| dx$ exists. Given a partition P of $[a, b]$, we have

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

This equals

$$\sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|}{|x_i - x_{i-1}|} |x_i - x_{i-1}|,$$

but $\frac{|f(x_i) - f(x_{i-1})|}{|x_i - x_{i-1}|}$ is $|f'(y_i)|$ for some $y_i \in [x_{i-1}, x_i]$, by the Mean Value Theorem. Therefore,

$$V(f, P) = \sum_{i=1}^{n} |f'(y_i)|(x_i - x_{i-1})$$

is a Cauchy sum for $|f'|$ on the partition $[a, b]$. Since $|f'|$ is integrable on $[a, b]$, we have that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions P with $|P| < \delta$ and all Cauchy sums $S(f, P)$, we have

$$|S(f, P) - \int_a^b |f'(x)| dx| < \varepsilon.$$

In particular,

$$|V(f, P) - \int_a^b |f'(x)|dx| < \varepsilon,$$

and we conclude that $V_a^b(f)$ exists and equals $\int_a^b |f'(x)|dx$.