4130 HOMEWORK 1

Due Thursday February 4

(1) Section 1.1.3 Exercise 2b.

"The only even prime is 2." There are many different ways of approaching the problem. One way is

$$\forall n \in \mathbb{N}(n \text{ is even} \wedge n \text{ is prime} \implies n = 2).$$

The negation is

$$\exists n \in \mathbb{N}(n \text{ is even} \wedge n \text{ is prime} \wedge n \neq 2).$$

That is, "There exists an even prime which is not equal to 2."

(2) Section 1.1.3 Exercise 3b.

"Every nonzero rational number has a rational reciprocal."

$$\forall x \in \mathbb{Q} \setminus \{0\} \exists y \in \mathbb{Q}(xy = 1).$$

The corresponding statement with quantifiers reversed is:

$$\exists y \in \mathbb{Q} \forall x \in \mathbb{Q} \setminus \{0\}(xy = 1).$$

This is false, because if $y \in \mathbb{Q}$ is such that $yx = 1$ for all $x \in \mathbb{Q} \setminus \{0\}$ then $y = 2y = 1$ which is impossible.

(3) Let A be a set and let $P(a)$ be a statement about an element of a. We write

$$\exists! a \in A\, P(a)$$

for "there exists a unique $a \in A$ such that $P(a)$".

(a) Write the statement $\exists! a \in A\, P(a)$ in a form which uses the quantifiers $\forall$ and $\exists$, and no connectives apart from $\wedge$, $\vee$ and $\neg$.

It can be written as

$$\exists a \in A(P(a) \wedge \forall b \in A(\neg P(b) \vee b = a)).$$

(b) Write the negation of the statement from part (a).

$$\forall a \in A(\neg P(a) \vee \exists b \in A(P(b) \wedge b \neq a)).$$

(4) Section 1.2.3 Exercise 2.

The set of all finite subsets of $\mathbb{N}$ is countable.

Proof: Let A denote the set of all finite subsets of $\mathbb{N}$. We need to define an injection $f : A \to \mathbb{N}$. Let $p_1 < p_2 < p_3 < \cdots$ denote the prime numbers listed in increasing order. Given a finite subset $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{N}$, relabel the x_i if necessary so that $x_1 < x_2 \cdots < x_n$. Then define

$$f(S) = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}.$$

By uniqueness of the decomposition of a natural number into a product of primes, if $S \neq T$ then $f(S) \neq f(T)$. Thus f is injective, as required.

(5) Section 1.2.3 Exercise 4.

Proof: Let A be an uncountable set. Let $C \subset A$ be countable. Suppose for a contradiction that $A \setminus C$ is countable. Then

$$A = (A \setminus C) \cup C$$

is a union of two countable sets. But, by a theorem from the lectures, a countable union of countable sets is countable. Thus, A is countable. This is a contradiction.

Therefore, $A \setminus C$ is uncountable.

(6) Section 1.2.3 Exercise 7. Let A be an infinite set. We wish to show that $|\mathcal{P}(A)| > |A|$. First, we show that there is an injection $A \to \mathcal{P}(A)$. This is clear, since we can map $x \in A$ to $\{x\} \in \mathcal{P}(A)$.

Next, we must show that there is no bijection $A \to \mathcal{P}(A)$. Suppose $f : A \to \mathcal{P}(A)$ is a bijection. Let

$$Z = \{a \in A : a \notin f(a)\}.$$

Since f is surjective, there exists $b \in A$ with $f(b) = Z$. Either $b \in f(b)$ or $b \notin f(b)$. If $b \in f(b)$ then $b \notin Z$ by definition of Z. But this contradicts $b \in f(b) = Z$. On the other hand, if $b \notin f(b)$ then $b \in Z$ by definition of Z. But then $b \in Z = f(b)$, a contradiction. So in either case, we get a contradiction. Therefore, the bijection f does not exist.