MATH 413 HONORS INTRODUCTION TO ANALYSIS I
PRELIM 1.
PRACTICE

(Note: attempt all questions. You have 70 minutes. Good luck!)

(1) **(9 marks)** Let $X = (0,1) \cup (2,3) \subset \mathbb{R}$. State whether the following statements about X are true or false and give a brief reason in each case.

(a) $3 \in \mathbb{R}$ is a cluster point (a.k.a. limit-point) of X.

Answer: True. For every $1/n$*, the intersection* $(3-1/n, 3+1/n) \cap X$ *is an open set and therefore contains infinitely many points of* X*. So* 3 *is a cluster point.*

(b) X is a closed set.

Answer: False. Since 3 *is a cluster point of* X *but* $3 \notin X$*,* X *cannot be closed.*

(c) The set $f^{-1}(X)$ is open, where $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^5 + 2x^3 - 9x + 1$.

Answer: True. The function f *is continuous, since it is a polynomial function. Its domain is* $\mathbb{R}$*, which is open. The set* X *is open. Therefore,* $f^{-1}(X)$ *is an open set.*

(2) **(25 marks)** Let $\{x_n\}$ be a sequence of real numbers.

(a) **(3 marks)** Define what it means for $\{x_n\}$ to converge to a limit $L \in \mathbb{R}$.

Answer: $\{x_n\}$ *converges to* L *if for all* $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that if* $n > N$ *then* $|x_n - L| < \varepsilon$.

(b) **(10 marks)** Show that if $\{x_n\}$ converges to the limits $L \in \mathbb{R}$ and to $M \in \mathbb{R}$ then $L = M$.

Answer: Let $\{x_n\}$ *be a sequence which converges to* L *and to* M*. Let* $\varepsilon > 0$*. Then there exists* $N_1 \in \mathbb{N}$ *such that if* $n > N_1$ *then* $|x_n - L| < \varepsilon$ *and there exists* $N_2 \in \mathbb{N}$ *such that if* $n > N_2$ *then* $|x_n - M| < \varepsilon$*. Therefore by the triangle inequality, if* $n > \max\{N_1, N_2\}$*, we get*

$$|L - M| \leq |L - x_n| + |x_n - M| < 2\epsilon. \qquad (*)$$

By the Archimedian property, if $L \neq M$ then there exists $t \in \mathbb{N}$ with $1/t < |L - M|$. But then if we take $\varepsilon < 1/2t$, we get a contradiction to $(*)$. So $L = M$.

(c) **(6 marks)** Let $a < b$. Prove the following theorem using any method you wish:

Theorem: If $\{x_n\}$ is a monotonically increasing sequence of points in $(a, b]$, then $\{x_n\}$ converges to a point of $(a, b]$.

Answer: Let $\{x_n\}$ be a monotonically increasing sequence of points with $x_n \in (a, b]$ for all n. By a theorem from the lectures, every monotonically increasing sequence of real numbers which is bounded above converges. The sequence $\{x_n\}$ is bounded above by b, so it converges to a real number x. We must show $x \in (a, b]$. One way to do this is to observe that since $a < x_1 \leq b$, we have $x_n \in [x_1, b]$ for all n. So $\{x_n\}$ is a convergent sequence of points in the closed set $[x_1, b]$ and therefore its limit x must also belong to $[x_1, b] \subset (a, b]$. So $x \in (a, b]$ as required.

(d) **(6 marks)** Show that the converse of the theorem in part (c) is false.

Answer: Recall that the converse of a statement $A \Rightarrow B$ is the statement $B \Rightarrow A$. So the converse states that if $\{x_n\}$ is a sequence of points in $(a, b]$ which converges to a point of $(a, b]$, then $\{x_n\}$ is monotonically increasing. But there are lots of sequences which converge to a point of $(a, b]$ but are not monotonically increasing. For example, choose c with $a < c < b$ and choose $t \in \mathbb{N}$ with $c + 1/t < b$. Then the sequence $\{c + \frac{1}{t+n}\}$ converges to c and is not monotonically increasing.

(3) **(16 marks)** Here is an unfinished proof of the following theorem:

Theorem: If $\{x_n\}$ and $\{y_n\}$ are bounded sequences then

$$\liminf\{x_n + y_n\} \geq \liminf\{x_n\} + \liminf\{y_n\}.$$

Proof: Let $n \in \mathbb{N}$. For each $t > n$, we have $x_t \geq \inf_{k>n}\{x_k\}$ and $y_t \geq \inf_{k>n}\{y_k\}$.

Therefore, $x_t + y_t \geq \inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}$. Therefore, the number $r = \inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}$ is a lower bound for the set $\{x_t + y_t : t > n\}$...

(a) **(9 marks)** Finish the proof of the theorem.

Answer: ... and therefore, since $\inf_{t>n}\{x_t + y_t\}$ *is the greatest lower bound for the set* $\{x_t + y_t : t > n\}$, *we have*

$$\inf_{t>n}\{x_t + y_t\} \geq r = \inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}.$$

This holds for every n *and therefore since non-strict inequalities are preserved by limits, we get*

$$\lim_{n\to\infty}\inf_{t>n}\{x_t + y_t\} \geq \lim_{n\to\infty}(\inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}).$$

(Here, we used the fact that the limits must exist since we know that $\{x_n\}$ *and* $\{y_n\}$ *(and hence* $\{x_n+y_n\}$*) are bounded sequences and therefore they have a finite* $\limsup$ *and* $\liminf$*.) Now we use the fact that the limit of a sum of convergent sequences is the sum of the limits, to get*

$$\lim_{n\to\infty}\inf_{t>n}\{x_t + y_t\} \geq \lim_{n\to\infty}\inf_{k>n}\{x_k\} + \lim_{n\to\infty}\inf_{k>n}\{y_k\}).$$

Finally, we use the theorem that the $\liminf$ *of a sequence* $\{a_n\}$ *is* $\lim_{n\to\infty}\inf_{k>n}\{a_k\}$, *to obtain*

$$\liminf\{x_n + y_n\} \geq \liminf\{x_n\} + \liminf\{y_n\}.$$

(b) **(7 marks)** Give an example of bounded sequences $\{x_n\}$ and $\{y_n\}$ such that $\liminf\{x_n + y_n\} \neq \liminf\{x_n\} + \liminf\{y_n\}$.

Answer: There are many possible answers. For example, let $\{x_n\}$ *be the sequence* $\{(-1)^n\}$ *and let* $\{y_n\}$ *be the sequence* $\{(-1)^{n+1}\}$. *Then* $x_n + y_n = 0$ *for all* n, *so* $\liminf\{x_n + y_n\} = 0$. *But* $\liminf\{x_n\} = \liminf\{y_n\} = -1$, *so*

$$0 = \liminf\{x_n + y_n\} \neq \liminf\{x_n\} + \liminf\{y_n\} = -2.$$

[**END OF PAPER**]