MATH 413 HONORS INTRODUCTION TO ANALYSIS I
PRELIM 1.
SOLUTIONS

(Note: attempt all questions. You have 70 minutes. Good luck!)

(1) **(9 marks)** Let $X = [0, 1] \cup \{3\} \subset \mathbb{R}$. State whether the following statements about X are true or false and give a brief reason in each case.

(a) X is bounded.

Answer: True. For all $x \in X$, $|x| \leq 3$, so X is a bounded set.

(b) X can be written as an intersection of countably many open sets.

Answer: True. For example,

$$X = \bigcap_{n=1}^{\infty} \left((-1/n, 1 + 1/n) \cup (3 - 1/n, 3 + 1/n)\right).$$

(c) There is a point $x_0 \in X$ at which the function $f(x) = x^4 - 3x^2 + 4$ achieves its infimum on X (that is, $f(x_0) = \inf\{f(x) : x \in X\}$).

Answer: True. Since X is a closed and bounded set, it is compact. The given function f is continuous, being a polynomial function, and so f achieves its infimum on X, by a theorem from class.

(2) **(25 marks)** Let $\{x_n\}$ be a sequence of real numbers.

(a) **(3 marks)** Define what it means for $\{x_n\}$ to converge to a limit $L \in \mathbb{R}$.

Answer: $\{x_n\}$ converges to L if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that if $n > N$ then $|x_n - L| < \varepsilon$.

(b) **(10 marks)** Show that if $\{x_n\}$ converges, then $\{x_n\}$ is bounded.

Answer: Suppose $\{x_n\}$ is a convergent sequence of real numbers. We need to show that there is a real number B such that $|x_n| \leq B$ for all $n \in \mathbb{N}$. Let $L = \lim_{n\to\infty} x_n$. Taking $\varepsilon = 1$ in the definition of convergence, we see that there exists $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - L| < 1$. By the triangle inequality,

$$|x_n| \leq |x_n - L| + |L| \leq 1 + |L|$$

if $n > N$. Now let $B = \max\{|x_1|, |x_2|, \ldots, |x_n|, |L|+1\}$. Then if $n < N$, we have $|x_n| \leq B$, and if $n > N$ then $|x_n| < |L|+1 \leq B$. So for all $n \in \mathbb{N}$, $|x_n| \leq B$ and therefore $\{x_n\}$ is a bounded sequence.

(c) **(6 marks)** Prove the following theorem using any method you wish:

Theorem: If $\{x_n\}$ converges to L then $\{x_n^4 - 3x_n^2 + 4\}$ converges to $L^4 - 3L^2 + 4$.

Answer: The easiest way to do this is to observe that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 + 3x^2 + 4$ is a continuous function. Therefore, if $\{x_n\}$ converges to L then $\{f(x_n)\}$ converges to $f(L)$, as required.

(d) **(6 marks)** Show that the converse of the theorem in part (c) is false.

Answer: The converse is the statement that if $\{x_n\}$ is a sequence of real numbers and $\{x_n^4 - 3x_n^2 + 4\}$ converges to $L^4 - 3L^2 + 4$, then $\{x_n\}$ converges to L. This is not true. For example, take $x_n = -1$ for all n, and take $L = 1$.

(3) **(16 marks)** Sally took an analysis exam and in the final question was asked to prove the following theorem:

Theorem. *If $s = \sup\{x \in \mathbb{Q} : x^2 < 2\}$ then $s^2 \geq 2$.*

Her proof began as follows:

Proof: Suppose for a contradiction that $s^2 < 2$. Let $\varepsilon = 2 - s^2 > 0$. By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{2s}{n} < \varepsilon/2$. Choose such an n which is large enough so that $\frac{1}{n^2} < \varepsilon/2$. Then $\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \varepsilon \ldots$

(a) **(9 marks)** Unfortunately, Sally ran out of time here. Finish her proof of the theorem.

Answer: $\ldots = 2$. So $\left(s + \frac{1}{n}\right)^2 < 2$. Now, by a theorem from the assignments, there exists $q \in \mathbb{Q}$ with $s < q < s + \frac{1}{n}$. Therefore, $s^2 < q^2 < \left(s + \frac{1}{n}\right)^2 < 2$. So $q \in \{x \in \mathbb{Q} : x^2 < 2\}$ and $q > s$. This contradicts the fact that s is supposed to be an upper bound for the set $\{x \in \mathbb{Q} : x^2 < 2\}$. Therefore, we must have $s^2 \geq 2$.

(b) **(7 marks)** Prove that $s^2 = 2$.

Answer: One way to do this is to observe that for each $n \in \mathbb{N}$, there must be a point $x_n \in \mathbb{R}$ with $x_n \in \{x \in \mathbb{Q} : x^2 < 2\}$ and $s - \frac{1}{n} < x_n < s$ (indeed, if this were not the case then $s - \frac{1}{n}$ would be an upper bound for $\{x \in \mathbb{Q} : x^2 < 2\}$

which was less than s). The sequence $\{x_n\}$ converges to s, and $x_n^2 < 2$ for all n. Therefore, by properties of limits, we have

$$s^2 = (\lim_{n\to\infty} x_n)^2 = \lim_{n\to\infty} x_n^2 \leq 2.$$

So $s^2 \leq 2$, and we have shown above that $s^2 \geq 2$. Therefore, $s^2 = 2$.

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